Charity Auctions

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Abstract

We present a model of charity auctions in which all bidders receive a benefit from the host charity raising revenue. Bidding behavior reflects two conflicting incentives: bids may be inflated because of private benefits from charitable giving, or bids could be depressed by the public goods nature of auction revenue. We study first- and second-price auctions and all-pay auctions. Revenue equivalence is unbalanced whenever a bidder benefits from the charity collecting another bidder's money. All-pay and second-price auctions have higher expected revenue than first-price auctions. The revenue ranking of all-pay and second-price auctions depends on parameter values, but as the number of bidders becomes large the all-pay auction is more lucrative than either single-price format.

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1 Introduction

In a typical auction the only bidder who benefits from a sale is the winner, and any payments that bidders make to the auctioneer are perceived (by their sources) as pure losses. Charity auctions, in which a charitable organization sells an item in order to raise revenue for its cause, are different. When an organization like the American Cancer Society holds an auction, it is reasonable to assume that the attending bidders have two objectives: to win items that they value and to support the host charity. If this is the case, then each dollar raised by the charity provides a benefit to auction participants. Auction revenue may be interpreted as an (imperfectly) public good that is beneficial to all bidders regardless of the revenue's source. This raises interesting questions about equilibrium bidding strategies in charity auctions. Do bids rise because auction participants view their payments as "subsidized" by charitable sentiment, or do they instead fall because there are benefits from losing a charity auction? What auction format raises the most revenue for organizers of charity auctions?

Charity auctions are a common (and lucrative) way of raising funds. One of the longest-running auctions in the world, the annual wine sale hosted by the Hospices de Beaune, is a charity auction. The 2000 sale, the 140th auction organized by the Hospices, benefitted Burgundy-area charitable groups and raised almost $5 million in revenue.[1] An American counterpart of this auction is the annual Napa Valley Wine Auction, which was held for the 20th time in 2000 and generated $9.5 million in revenue. Musician Eric Clapton conducted a charity auction in 1999 that offered an ironic complement to these wine sales. Clapton, who struggled with alcohol and drug addiction during the 1970s, sold 100 of his guitars and raised $5 million for his substance abuse treatment facility on the Caribbean island of Antigua. Charity auctions are frequently used to raise funds for schools but these auctions usually do not collect extraordinary amounts of revenue. An exception is documented by David Kaplan [1999] in *The Silicon Boys and Their Valley of Dreams.* Woodside, California is a suburb of San Francisco that has a large population of computer industry multi-millionaires. When the region's public elementary school held a charity auction in 1998 the prizes included a week at NASA's Space Camp and a week-long cruise on Oracle founder Larry Ellison's yacht; over $400,000 was raised in a single night.

We present a model of charity auctions in which risk-neutral bidders have independently drawn

[1] In recent years the identity of the charity receiving assistance has changed with each auction. The Hospices' mission is to aid organizations that benefit the indigent and sick.

private values for a single auctioned item and each bidder receives a benefit from the charity's revenue. We permit bidders to benefit more strongly from their own payments than from those of other bidders, but the model is symmetric in all other respects. Three selling formats are considered: first- and second-price auctions and all-pay auctions.[2]

We find that bidders in a first-price charity auction are more aggressive than in a standard (non-charity) auction because of the benefit they receive from their own payments to the charity when they win. A similar effect exists in second-price charity auctions, plus there is the possibility that a bidder submits the second-highest tender and determines the payment of the winner. This additional incentive to increase one's bid in a second-price charity auction unbalances the revenue equivalence result first presented by Vickrey [1961] and later generalized by Myerson [1981]. Second-price charity auctions are more lucrative than first-price charity auctions.

When a bidder in a first- or second-price auction increases her bid, she decreases the probability that another bidder will win the auction and make a payment to the auctioneer. In a charity auction other bidders' payments to the auctioneer are valuable, and this may depress bids in single-price auctions. All-pay charity auctions do not have this characteristic; increasing one's bid in an all-pay auction does not affect the chance that other bidders will pay the auctioneer. We demonstrate that all-pay auctions have higher expected revenue than both single-price auctions when the number of bidders is large. The all-pay auction is always more lucrative than a first-price auction, but it is not possible to provide a general revenue ranking for all-pay and second-price auctions.

Although to our knowledge charity auctions have not been considered previously in the literature, there are several other auction settings in which transfers to losing bidders influence bidding incentives. "Knockout" auctions that determine allocations among members of a cartel typically include a payment to auction participants who do not win the knockout auction. Graham and Marshall [1987] and McAfee and McMillan [1992] characterize optimal mechanisms for allocating auction profits among members of a cartel.[3] Engelbrecht-Wiggans [1994] studies auctions with benefits to auction participants that are proportional to the winning bid. The motivating example

[2] In an all-pay auction, all bidders pay their bids and the person with the highest bid receives the item for sale.

[3] Both papers include payments to losing participants in a knockout auction, but the mechanisms do not provide each bidder with a positive benefit that is proportional to the amount paid (as in a charity auction). In Graham and Marshall [1987] all n participants in the knockout auction receive the same payment *before* bids are submitted in the knockout round. In McAfee and McMillan [1992] the losers of the knockout auction each get $\frac{1}{n-1}$ of the difference between the winner's payment and a reservation price; the winner's utility is the difference between her valuation and her payment.

used by Engelbrecht-Wiggans is an estate auction in which bidders (the children of the deceased) assign ownership of a family farm to the highest bidder, but all bidders collect an equal share of the auction revenue. As in a charity auction, bidders have an incentive to inflate their tenders in order to improve their return from losing the estate auction. We extend Engelbrecht-Wiggans' model by allowing a more general structure of benefits to auction participants and by considering all-pay auctions in addition to first- and second-price formats.[4] Another type of auction in which a bidder may receive a payment is a corporate takeover, in which a bidder with partial ownership of the firm for sale (a "toehold") receives a portion of the winner's payment. Burkart [1995] and Singh [1998] consider the effect of toeholds on bidding in a private-value auction; Bulow, Huang, and Klemperer [1999] demonstrate that bidders' incentives and sale prices in common-value takeover auctions can be substantially different from the private-value case.

Jehiel, Moldovanu, and Stacchetti [1996, 1999] consider situations in which an auction winner imposes an externality on losing bidders. Auction participants hold private information regarding their own value of the object for sale and their valuations of auction outcomes in which they do not receive the object. Payments to a charity may also be interpreted as affecting all bidders with a positive externality, but the magnitudes of the externalities studied by Jehiel, Moldovanu, and Stacchetti are unaffected by bidding intensity. Budget-constrained bidders may also impose externalities on each other when participating in a sequence of auctions. Bidding is aggressive in an early auction because a high price weakens the ability of the early winner to succeed in later auctions. Pitchik and Schotter [1988] and Benoît and Krishna [2001] study this phenomenon for the case of common values over the auctioned items and complete information about valuations and budgets.

The description of auction revenue as a public good invites comparison between the present research and other studies of funding public goods. When contributions to a pure public good are voluntary the public good generally will be under-provided and contributions from one source "crowd out" other donations dollar-for-dollar. Our model of charity auctions ensures that bidders always have a private gain from winning (and therefore participating in) the contest, but bidding may become less aggressive as the public goods effect from others' payments becomes more valuable. Personal gains from donations to a public good have been considered by Andreoni [1989] and

[4] During the course of this research we found that Engelbrecht-Wiggans' comparison of expected revenue from first- and second-price auctions applies only to cases in which there are two bidders. In light of this, our analysis of revenue from single-price auctions with more than two participants is also novel.

Morgan [2000]. Andreoni describes giving to a public good when donors receive an egotistical warm glow from their own contributions. Like Andreoni, we find that a warm glow can increase revenue. Morgan studies the use of lotteries to raise revenue for a public good. As in an auction, all lottery participants have a chance to win a prize for personal gain, and the combination of private and public gains from ticket purchases increases revenue relative to the case with voluntary donations directly to the public good. A key difference between Morgan's lottery model and our charity auction model is that all of the lottery contestants value the prize equally, so there are no (in)efficiency consequences of an allocation mechanism that is random.[5]

2 Preliminaries

Suppose a charitable organization possesses one item to sell at an auction, and all of the revenue from this sale will support the charity's mission. The organization considers three sealed-bid auction formats: first-price, second-price, and all-pay. For analytic simplicity we assume that the charity does not implement revenue-enhancing strategies like minimum bids and entry fees.

There are $n \geq 2$ risk-neutral bidders who draw independently from the probability distribution F with support $[\underline{t}, \overline{t}]$ to determine their private valuation for the auctioned item. An individual's taste, t, for the item is private information, but distribution F is common knowledge. Let F be differentiable on the interior of its support with positive density f.

Individual surplus from the charity auction has both private and philanthropic components. As in the standard independent private values auction model, an individual receives surplus from private consumption when the object is purchased at a price less than her valuation. Additional surplus comes to the individual through the host charity collecting revenue for a good cause. Moreover, we follow Andreoni [1989] and allow for the possibility of a warm glow to an individual – the additional satisfaction that the money going to the charitable organization is her own.

We specify that the psychological return to person i from the host charity collecting one dollar from i is θ, while the return to i from another bidder's dollar being transferred to the charity is λ. To allow for a warm glow from charitable giving, we specify that the parameters θ and λ satisfy $0 \leq \lambda \leq \theta < 1$. The parameter θ is restricted to be less than one because values of $\theta \geq 1$ render the idea of a charity auction moot since i wishes to transfer all of her money to the charity. If

[5] The lottery contestant who purchases the most tickets has the best chance of winning the prize, but other contestants may win instead.

$\theta = \lambda$, the psychological return from participating in the auction is purely altruistic; individual i's satisfaction from seeing money go to a favored cause is independent of the source of the revenue. When $\theta > \lambda$, there is an additional, egotistical benefit to person i when it is her own money that goes to the charity. We will find it convenient later to denote this warm glow by $\Delta = \theta - \lambda$.

3 Bidding in a Charity Auction

We derive equilibrium bid functions for the three auction formats listed in the previous section. The solution technique is to consider the signalling problem of person i given that all other participants in the auction use the same strictly increasing function B to map their own valuations of the auctioned item into bids.[6] We assume that B is strictly increasing and differentiable on the interval $(\underline{t}, \overline{t})$. Bidder i is not obliged to implicitly announce her true type, t. She may select a valuation s from $[\underline{t}, \overline{t}]$ and submit a bid of $B(s)$. There is no need for i to consider bids outside of the range $[B(\underline{t}), B(\overline{t})]$; doing so can never help i and may hurt her.[7] In a symmetric equilibrium i chooses to select the bid $B(t)$ using B and her own type t.

3.1 First-Price Auctions

In a first-price auction bidder i's return depends on her type, her bid, and the highest bid made by other bidders. Let x be the highest of the $(n-1)$ other bidders' valuations of the item being sold; the distribution of x is $F(x)^{n-1}$. If B_1 is the bid function used by the other bidders in a first-price auction, the highest bid made by the others is $B_1(x)$. Suppose bidder i imitates a person with type s and bids $B_1(s)$. We divide the support of x into two regions, above and below s. If $s > x$ bidder i pays $B_1(s)$ to the charity and receives the auctioned item (worth t to i) and a psychological return of $\theta B_1(s)$. When $s < x$ bidder i receives λ times the payment made by the auction winner

[6] Alternatively, we could describe i's problem as one of choosing an arbitrary bid, b_i, while the other bidders use the function B. This would not affect our results on equilibrium bidding strategies or revenue rankings. We adopt the signalling approach for notational ease.

[7] In all three auction formats a bid above $B(\overline{t})$ cannot improve i's probability of victory (which is one at $B(\overline{t})$) but may result in an unnecessarily high payment to the auctioneer. A bid of $B(\underline{t})$ or below implies that i will lose the auction with certainty. The reasons why bids below $B(\underline{t})$ (when feasible) do not make i better off depend on the rules of each auction; this should be apparent to the reader after the equilibrium bid functions are derived in Section 3.

to the charity. In total, the expected return to i from imitating a bidder of type s is:

$$\pi(s|t,n) = \int_{\underline{t}}^{s} [t-(1-\theta)B_1(s)]dF(x)^{n-1} + \lambda \int_{s}^{\bar{t}} B_1(x)dF(x)^{n-1}. \tag{1}$$

To further consider the effect of θ and λ on bidding, divide (1) into the sum of a bidder's expected return from a standard (non-charity) auction and a payoff function, Φ_1, that includes all charity-related effects:

$$\begin{aligned} \pi(s|t,n) &= F(s)^{n-1}[t-B_1(s)] + \Phi_1(s), \\ \text{with } \Phi_1(s) &= \theta F(s)^{n-1}B_1(s) + \lambda \int_{s}^{\bar{t}} B_1(x)dF(x)^{n-1}. \end{aligned}$$

If bidder i selects a value of s while ignoring the terms collected in Φ_1, she faces the standard trade-off between increasing her chance of winning the auction and increasing her expected payment. The terms in Φ_1 provide i with an additional incentive to increase her choice of s, given that other bidders use the fixed bid function B_1. When the bid function is strictly increasing and $\theta \geq \lambda$, Φ_1 is increasing in a bidder's choice of s:

$$\Phi_1'(s) = \theta B_1'(s)F(s)^{n-1} + (\theta-\lambda)B_1(s)\frac{dF(s)^{n-1}}{ds}.$$

If a bidder increases her choice of s by a small amount, she directly benefits by θ times the increase in her expected payment. Additionally, the increased chance of i's own payment upon winning the auction is at least as valuable as the reduced probability of another bidder's payment because of the warm glow ($\theta \geq \lambda$) from transfers to the charity. Together, these points suggest that bidding in a first-price charity auction is more aggressive than in an auction with $\Phi_1 = 0$.[8]

In a symmetric equilibrium, an incentive-compatible bidding strategy implies that it is optimal for i to select $s = t$ (her own type). Regardless of auction format, the first-order condition for an incentive-compatible selection of s is

$$\left.\frac{\partial \pi(s|t,n)}{\partial s}\right|_{s=t} = 0. \tag{2}$$

When (2) is applied to the expected return given in (1) for a first-price charity auction, we obtain

$$[t-(1-\theta+\lambda)B_1(t)]\frac{dF(t)^{n-1}}{dt} - (1-\theta)B_1'(t)F(t)^{n-1} = 0. \tag{3}$$

[8]This does not prove that bidding in a charity auction is more aggressive than in a standard auction. We have demonstrated that the charity component of this auction leads one bidder to increase her selected value of s, holding fixed the bidding strategies of the other auction participants. We still must verify that bidding is more aggressive in equilibrium.

Recall that $\Delta = \theta - \lambda$, and rearrange (3) to obtain the differential equation

$$B_1'(t) + B_1(t)\left(\frac{1-\Delta}{1-\theta}\right)\frac{\frac{d}{dt}\left[F(t)^{n-1}\right]}{F(t)^{n-1}} = \frac{t}{1-\theta}\frac{\frac{d}{dt}\left[F(t)^{n-1}\right]}{F(t)^{n-1}}. \tag{4}$$

The term $\frac{(n-1)(1-\Delta)}{(1-\theta)}$ would appear frequently in the discussion below, so it is replaced by α for simplicity. To solve the differential equation (4) we multiply each side of the expression by the integrating factor $F(t)^{\alpha}$ to obtain

$$\frac{d}{dt}[B_1(t)F(t)^{\alpha}] = \frac{t}{1-\Delta}\frac{d}{dt}[F(t)^{\alpha}].$$

Integrating from $\underline{t}$ to t (along with the boundary condition $B_1(\underline{t})F(\underline{t})^{\alpha} = 0$) yields the bid function for a first-price charity auction:

$$B_1(t) = \frac{1}{1-\Delta}\left[t - \int_{\underline{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{\alpha} dx\right]. \tag{5}$$

The derivation above establishes that this function is an equilibrium bidding rule if it is increasing in t, differentiable, and a maximal solution to the bidder's signaling problem rather than a minimum. We verify that B_1 (and all other bid functions derived below) satisfies these conditions in the appendix. The derivation also establishes that B_1 is the unique equilibrium among all possible symmetric, increasing, and differentiable bidding rules for bidders with valuations in $(\underline{t}, \overline{t}]$. The strategy for a bidder with the lowest valuation $\underline{t}$ is indeterminate – any bid between 0 and $\frac{\underline{t}}{1-\Delta}$ is optimal for a person with the valuation $\underline{t}$. A similar indeterminacy at boundary points of t's support can occur in the second-price charity auction too. However, these issues do not affect the expected revenue of either auction because the indeterminacy is restricted to subsets of the support with zero probability measure.

If θ and λ are zero, B_1 becomes the bid function for a first-price auction with independent private values and no charity component:

$$B_1(t) = t - \int_{\underline{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{n-1} dx \qquad (\text{when } \ \theta, \lambda = 0).$$

When there is only an egotistical return from charitable giving ($\theta > 0$, $\lambda = 0$), the parameter θ behaves exactly like a subsidy to i's payment. The bidding function used in a standard auction is simply inflated through multiplication by $(1-\theta)^{-1}$. The other borderline case in a charity auction is that of purely altruistic behavior ($\theta = \lambda > 0$). In this situation, the bid function $B_1(t)$ leads to higher bids than in the non-charity case because bidder i behaves as if she has more than $(n-1)$ competitors for the auctioned item. That is, if we solve $m - 1 = \frac{n-1}{1-\theta}$ for m, we can say that bidders behave as if there are $m = \frac{n-\theta}{1-\theta}$ potential buyers in a standard auction rather than n.[9] When $\theta = \lambda$ and both parameters approach one, a fixed, finite number of bidders in a charity auction behave as if they are in a standard auction with an infinite number of bidders.

Intuition suggests that the bidding function $B_1(t)$ may decrease as the parameter λ increases. This "crowding-out" effect reduces one's own bidding when the attractiveness of successful bids by others increases. The effect of an increase in λ on B_1 is

$$\frac{\partial B_1(t)}{\partial \lambda} = -\frac{1}{(1-\Delta)^2}\left\{t - \int_{\underline{t}}^{t}\left[1-\alpha\log\left(\frac{F(x)}{F(t)}\right)\right]\left(\frac{F(x)}{F(t)}\right)^{\alpha}dx\right\}. \tag{6}$$

Although the sign of (6) may appear to be ambiguous because the term in braces has both positive and negative components, we find that (6) is always negative. First, notice that

$$\left.\frac{\partial B_1(t)}{\partial \lambda}\right|_{t=\underline{t}} = -\frac{\underline{t}}{(1-\Delta)^2} \leq 0.$$

Next, note that the cross-partial derivative

$$\frac{\partial^2 B_1(t)}{\partial \lambda \partial t} = \left(\frac{\alpha}{1-\Delta}\right)^2 \int_{\underline{t}}^{t} \log\left(\frac{F(x)}{F(t)}\right)\left(\frac{F(x)}{F(t)}\right)^{\alpha}\frac{f(t)}{F(t)}dx.$$

is always negative. Since $\frac{\partial B_1(t)}{\partial \lambda}$ is negative at its lower boundary and decreasing in t, it must be the case that $\frac{\partial B_1(t)}{\partial \lambda}$ is negative for all t. There is crowding-out in first price charity auctions when the payments of bidders other than i yield a positive benefit to i.

3.2 Second-Price Auctions

In a second-price charity auction the payoff to bidder i depends on her type, her bid, and the highest and second-highest bids submitted by the other participants in the auction. Suppose x is a random variable that may represent the highest or second-highest of the other $(n-1)$ bidders' valuations. Again, if x is the highest type its distribution is $F(x)^{n-1}$. If x represents the second-highest type it has the distribution $\{(n-1)F(x)^{n-2}[1-F(x)] + F(x)^{n-1}\}$. All bidders use the function B_2 to map their (implicitly announced) tastes for the auctioned item into bids. The possible outcomes for i in a second-price charity auction can be divided into three cases. First, i wins the auction because she signals a type, s, that is greater than the highest (and second-highest) types of the $(n-1)$ other bidders. This case results in i paying the highest bid submitted by the other auction participants. In return for her payment i receives the prize (worth t) and a psychological benefit

[9] When $\theta = \lambda$, the first-order condition (3) can be multiplied by F^{m-n} to obtain the optimality condition for bidding in a non-charity auction with $\frac{n-\theta}{1-\theta}$ bidders.

of θ for each dollar she pays the auctioneer. Second, with probability $(n-1)F(s)^{n-2}[1-F(s)]$ person i submits the second-highest bid. Since the winner of the auction pays the second-highest bid, i receives a return of $\lambda B_2(s)$ on auction revenue equal to $B_2(s)$. Third, the type selected by i is smaller than the first- and second-highest types of the other $(n-1)$ bidders. The auction winner pays the second-highest bid and i's return is λ for each dollar of auction revenue. Combining these three cases, we write the expected return to bidder i with type t from imitating a person of type s as:

$$\begin{aligned}\pi(s|t,n) &= \int_{\underline{t}}^{s}[t-(1-\theta)B_2(x)]dF(x)^{n-1}+\lambda(n-1)F(s)^{n-2}[1-F(s)]B_2(s)\\ &\quad +\lambda\int_{s}^{\bar{t}}B_2(x)(n-1)(n-2)F(x)^{n-3}[1-F(x)]dF(x).\end{aligned}$$

Again, we split i's expected return from the auction into a non-charity component and $\Phi_2(s)$, the bidder's charity-related surplus when she mimics a person with valuation s:

$$\begin{aligned}\pi(s|t,n) &= \int_{\underline{t}}^{s}[t-B_2(x)]dF(x)^{n-1}+\Phi_2(s),\\ \text{with } \Phi_2(s) &= \theta\int_{\underline{t}}^{s}B_2(x)dF(x)^{n-1}+\lambda(n-1)F(s)^{n-2}[1-F(s)]B_2(s)\\ &\quad +\lambda\int_{s}^{\bar{t}}B_2(x)(n-1)(n-2)F(x)^{n-3}[1-F(x)]dF(x).\end{aligned}$$

The term Φ_2 is increasing in s, suggesting that participants in a charity auction bid more aggressively than in a standard auction:

$$\Phi_2'(s)=\lambda(n-1)F(s)^{n-2}[1-F(s)]B_2'(s)+(\theta-\lambda)B_2(s)\frac{dF(s)^{n-1}}{ds}.$$

The first term in Φ_2' is the increase in expected surplus from placing second with slightly higher bid. The second term accounts for the increased probability of i winning the auction and the decreased chance that i places third or lower. Note the similarity of the second terms in Φ_2' and Φ_1'. The lost surplus from reducing the chance that another bidder makes a payment to the auctioneer is effectively replaced by i's own, more valuable increased chance of winning.

Applying the incentive-compatibility condition (2), which holds in any type of auction, we obtain the first-order condition

$$[t-(1-\Delta)B_2(t)]\frac{dF(t)^{n-1}}{dt}+\lambda(n-1)F(t)^{n-2}[1-F(t)]B_2'(t)=0. \tag{7}$$

Simplifying (7) yields the differential equation

$$B_2'(t)-B_2(t)\left(\frac{1-\Delta}{\lambda}\right)\left(\frac{f(t)}{1-F(t)}\right)=-\frac{t}{\lambda}\left(\frac{f(t)}{1-F(t)}\right) \tag{8}$$

Let β denote $\frac{1-\Delta}{\lambda}$ and multiply each side of (8) by the integrating factor $[1-F(t)]^{\beta}$ to obtain

$$\frac{d}{dt}\left\{B_2(t)[1-F(t)]^{\beta}\right\} = \left(\frac{t}{1-\Delta}\right)\frac{d}{dt}\{[1-F(t)]^{\beta}\}.$$

When we integrate both sides of this expression from t to $\bar{t}$ and use the boundary condition $B_2(\bar{t})[1-F(\bar{t})]^{\beta} = 0$ we obtain our bid function for a second-price charity auction:

$$B_2(t) = \left(\frac{1}{1-\Delta}\right)\left\{t + \int_t^{\bar{t}} \left(\frac{1-F(x)}{1-F(t)}\right)^{\beta} dx\right\} \tag{9}$$

As in a standard private values auction, the bid function for a second-price charity auction is independent of the number of bidders, n. Also notice that when θ and λ are zero B_2 simplifies to t, the well-known bidding rule for second-price independent private values auctions.[10] When $\theta > 0$ and $\lambda = 0$ (purely egotistical returns from charitable giving), θ affects B_2 as a subsidy would in a standard auction. When θ and λ are both positive we have $B_2(t) > B_1(t)\ \forall\ t$.

The derivation also establishes that the symmetric increasing bidding equilibrium is unique for all bidders whose types lie in the interior of the support of the distribution. If the number of bidders exceeds 2 then the bid for the lowest type is again indeterminate – any bid between 0 and the value from the above equation is optimal for type $\underline{t}$. Similarly, a bidder with the valuation $\bar{t}$ has an indeterminate bid, which can be any value that is at least the value given by the above expression, namely $B_2(\bar{t}) = \bar{t}/(1-\Delta)$. For simplicity we break the indeterminacy at the endpoints by assuming that the bid function is continuous at the endpoints of its support, but nothing essential relies on this supposition because expected revenue is unaffected by restrictions on a set of probability zero.

Bidders with valuations near the upper boundary of t's support submit bids that approach $\bar{t}/(1-\Delta)$. The role of λ in Δ implies that this person's bid is decreasing in λ. However, unlike the case of first-price charity auctions, bidding in a second-price auction is not always weakened by an increase in the value that bidders place on others' payments. The expected surplus from the charity component of the auction, Φ_2, contains one term that is increasing in λ and one term that decreases in λ. The effect of a marginal change in λ on the equilibrium bidding function is

$$\frac{\partial B_2(t)}{\partial \lambda} = -\left(\frac{1}{1-\Delta}\right)^2\left\{t + \int_t^{\bar{t}}\left[1 + \left(\frac{1-\theta}{\lambda}\right)\beta\log\left(\frac{1-F(x)}{1-F(t)}\right)\right]\left(\frac{1-F(x)}{1-F(t)}\right)^{\beta} dx\right\}.$$

The integrand in $\frac{\partial B_2}{\partial \lambda}$ may be positive or negative, so it is not possible conclude that this derivative has the same sign for all t. Intuition might suggest that $\frac{\partial B_2}{\partial \lambda}$ is at least positive for valuations

[10] A decrease in λ leads to an increase in β, and as λ goes to zero β explodes and $\left(\frac{1-F(x)}{1-F(t)}\right)^{\beta}$ disappears.

near $\underline{t}$, where bidders are more likely to determine the payment made by the winner than win the auction themselves, but this is not always true. Bidders with intermediate and high valuations will reduce their bidding as λ grows for the same reason that similar bidders do so in first-price auctions: losing the auction becomes more attractive. These bidders put downward pressure on the bids of people with low values of t (who do not want to out-bid competitors with higher t's in this separating equilibrium), so it is possible for all bids to fall with an increase in λ.

Some of these properties of B_2 are effectively illustrated with a relatively simple example.

Example 1 *Assume that the iid valuations of participants in a second-price charity auction are distributed uniformly on* $[\underline{t}, \overline{t}]$. *Then the equilibrium bid function is*

$$B_2^U(t) = \left(\frac{1}{1-\Delta}\right)\left(\frac{\overline{t}+\beta t}{1+\beta}\right).$$

In Figure 1 we illustrate the ambiguous effects of changes in λ on B_2. We assume that $t \sim U[0,1]$ and graph B_2^U for a variety of λ values, holding θ fixed at 0.7. $B_2^U(1)$ falls as λ increases, but $B_2^U(0)$ is higher when $\lambda = 0.2$ than it is for $\lambda = 0.0$ and 0.5. The apparent increase in $B_2^U(0)$ with λ when λ is relatively low does not generalize to the case of t distributed uniformly on the arbitrary bounded support $[\underline{t}, \overline{t}]$. When $t \sim U[\underline{t}, \overline{t}]$ we find:

$$\left.\frac{\partial B_2^U(t)}{\partial \lambda}\right|_{\lambda=0} = \left(\frac{1}{1-\theta}\right)^2 (\overline{t} - 2t).$$

As long as $\underline{t} > \overline{t}/2$, the value of this derivative is negative for all t.

<< INSERT FIGURE 1 APPROXIMATELY HERE >>

It is frequently the case that second-price sealed-bid auctions are studied because they are a convenient substitute for the more common open, ascending-bid (English) auction. Bidding incentives in the two auction formats are very similar in private-value non-charity settings, and expected revenue from the two auction formats is identical. We find that this parallel also exists for charity auctions. This relationship between the auctions is studied through the "button auction," a stylized version of the English auction. A central feature of an oral auction – the opportunity to observe contestants drop out of the bidding – is retained in the button auction.

The rules of a button auction are as follows. An auctioneer displays a continuously increasing price for the object being sold. Each bidder has a personal button, and at the beginning of the auction each bidder is pressing her own button. A depressed button means that the bidder is willing to buy the object at its current price. When the announced price is higher than the bidder is willing to pay, she releases her button to exit the auction. Once a bidder exits the auction she may not return. The auction is over when only one bidder is pressing her button; this bidder wins the object and pays the auctioneer the price that was posted when the second-highest bidder exited the auction.

In the proposition below we argue that the equilibrium bid function for second-price sealed-bid auction, B_2, is also an equilibrium "exit strategy" for contestants in a button auction. (All proofs are collected in the appendix.)

Proposition 1. *The following is a Perfect Bayesian Equilibrium in a button auction: Each bidder of type t who has not yet released her button will release her button at the announced price p if and only if $p \geq B_2(t)$. The outcome of the button auction in terms of allocations and payments is identical to the second-price sealed-bid auction. Moreover, these results for the button auction hold whether or not bidders observe when their rivals exit.*

An important consequence of this proposition is that bidders' strategies are completely unaffected by the opportunity to observe other contestants exit the auction. Despite the information released about some bidders' valuations when exits are observed, the uncertainty over the remaining bidders' tastes preserves the structure of individuals' decision problems as they existed before the contest began. Note how different the bidding strategies in the button auction would be if all valuations were common knowledge. In this full-information situation all bidders other than the one with the highest valuation would want to remain in the auction for as long as possible to inflate the payment that the winner makes to the charity.

3.3 All-Pay Auctions

In an all-pay auction the highest bidder wins the object but all auction participants must pay their bids. Negative bids are not permitted. As in a first-price auction, i's return depends on her type, her bid, and the highest offer tendered by the $(n-1)$ other bidders. Let x represent the valuation of the bidder other than i with the highest type; x is an order statistic with the distribution $F(x)^{n-1}$. All bidders use the function B_A to map valuations into bids. When i mimics an individual with type s she bids $B_A(s)$, and the highest bid submitted by the other auction participants is $B_A(x)$. If $B_A(s)$ is larger than $B_A(x)$ bidder i receives the prize, which she values at t, and i pays $B_A(s)$ to the auctioneer. Bidder i loses the auction when she imitates a type that is less than x, but she still must pay her bid of $B_A(s)$. Regardless of the identity of the winner, i receives a benefit of θ for each of her own dollars transferred to the auctioneer and λ for each dollar bid (and paid) by the other auction participants. The two possible auction outcomes are combined to yield i's expected return from an all-pay charity auction:

$$\pi(s|t,n) = \int_{\underline{t}}^{s} t dF(x)^{n-1} - (1-\theta)B_A(s) + \lambda(n-1)\int_{\underline{t}}^{\overline{t}} B_A(x)dF(x). \tag{10}$$

The all-pay structure of this auction is evident in that i's bid of $B(s)$ is paid regardless of the identity of the auction winner. An important feature of (10) is that i's choice of s does not affect her benefit from payments made by the $(n-1)$ other auction participants. When we separate π into non-charity and charity components,

$$\pi(s|t,n) = tF(s)^{n-1} - B_A(s) + \Phi_A(s)$$
$$\text{with } \Phi_A(s) = \theta B_A(s) + \lambda(n-1)\int_{\underline{t}}^{\overline{t}} B_A(x)dF(x),$$

we note that i's surplus from the auctioneer's revenue increases in her own reported type, s. Additionally, i's incremental surplus from increasing s is just $\Phi'_A = \theta B'_A(s)$, which is independent of λ and other bidders' payments.

As in single-price auctions, incentive compatibility is captured by condition (2). Differentiating (10), restricting $s = t$, and setting the result to zero yields

$$t\frac{dF(t)^{n-1}}{dt} - (1-\theta)B'_A(t) = 0. \tag{11}$$

The first-order condition in (11) can be rearranged into a simple differential equation:

$$\frac{d}{dt}\left[B_A(t)\right] = \frac{t}{1-\theta}\frac{dF(t)^{n-1}}{dt}. \tag{12}$$

We integrate (12) from $\underline{t}$ to t, employ the boundary condition $B_A(\underline{t}) = 0$, and find that the bid function for an all-pay charity auction is

$$B_A(t) = \frac{1}{1-\theta}\left[tF(t)^{n-1} - \int_{\underline{t}}^{t} F(x)^{n-1}dx\right]. \tag{13}$$

To understand why $B_A(\underline{t}) = 0$, suppose that the equilibrium bid function has $B_A(\underline{t}) > 0$. In the posited equilibrium a bidder with the lowest possible type has no chance of winning the auction but still makes a payment to the auctioneer. This implies that for type $\underline{t}$ the first term in (10) is zero and the second is negative. The bidder's return would increase with a reduction in $B_A(\underline{t})$, therefore no bid function with $B_A(\underline{t}) > 0$ is an admissible equilibrium bidding rule.

The function (13) differs from bid functions in private value all-pay auctions only in the term $\frac{1}{1-\theta}$. For the cases of first- and second-price charity auctions we found that an effect on bidding can occur from increases in λ. This is clearly not the case for the bid function in (13); λ is absent from the all-pay bid function. Although a bidder benefits from an increase in λ because her utility from others' payments rises, a change in λ does not affect the bidding incentives of auction participants.

4 Revenue Comparisons

We now compare expected revenue from the three auction formats analyzed in Section 3. We begin with a discussion of auction revenue when $\lambda = 0$. When bidders place no value on auction revenue from people other than themselves, the auction formats considered above are revenue-equivalent. Next, we describe the possible revenue rankings between all-pay auctions and first- and second-price auctions for $\lambda > 0$. The section concludes with a comparison of expected revenue from single-price auctions while λ is positive.

4.1 A Simple Revenue Equivalence Result

Suppose $\lambda = 0$, so that the charity auction is essentially a standard (non-charity) auction with an *ad valorem* subsidy of θ to each bidder. When participants in the charity auction do not benefit from other bidders' payments each participant inflates by $(1-\theta)^{-1}$ the bidding strategy that would be used in a standard auction. It is well known that non-charity all-pay, first-price, and second-price auctions have the same expected revenue when bidders' have independently drawn private values for the prize.[11] The simple inflation of bid functions by $(1-\theta)^{-1}$ does not affect revenue equivalence.

We derive this result (and establish some notation that is useful below) by examining properties of utility from bidding in a charity auction while $\lambda = 0$. First, note that when bidder i chooses to announce type s in the signalling game described above, her probability of winning any type of charity auction is $F(s)^{n-1}$. Second, we may write the expected payment of bidder i as $P(s, n)$, where P is a function that depends on the rules of the auction in which i is participating. These terms may be incorporated into an alternative expression for i's payoff from a charity auction when $\lambda = 0$:

$$\pi(s|t, n) = F(s)^{n-1}t + (1-\theta)P(s, n).$$

[11] See Klemperer [1999] for a detailed survey of revenue equivalence.

When i reports her type truthfully (and all other bidders follow their equilibrium strategies), we write expected utility as $U(t,n) = \pi(t|t,n)$, so that

$$U(t,n) = F(t)^{n-1}t - (1-\theta)P(t,n). \tag{14}$$

As bidder i's choice of s comes from the optimality condition (2), we may use the envelope theorem and write

$$\frac{\partial U(t,n)}{\partial t} = F(t)^{n-1}. \tag{15}$$

The slope of the utility function is thus independent of auction format. We have assumed that bidders receive no benefit from the payments made by others to the charity, so a bidder with the lowest possible type ($\underline{t}$) is sure to lose the auction and she will have zero utility regardless of auction format. This provides us with a boundary condition, $U(\underline{t}, n) = 0$, for the differential equation in (15). The solution to this differential equation provides an expression for expected utility that is independent of auction format but must be equal to (14). If (14) is independent of auction format for an arbitrary bidder i then the expected payment, P, of any bidder is also the same for any auction format. Since expected revenue is determined by the expected payments of bidders, we conclude that expected revenue is equal for any auction in which (15) holds and $U(\underline{t}, n) = 0$.

4.2 All-Pay v. Single-Price Auctions

Next we consider situations in which $\theta \geq \lambda > 0$. Recall that the bidding function for an all-pay charity auction is independent of λ, so the expected revenue from an all-pay auction with a fixed value of θ is unaffected by changes in λ. This is not the case for first-price auctions. In Section 3.1 we established that participants in first-price auctions bid less aggressively when λ increases ($\frac{\partial B_1}{\partial \lambda} < 0$). Expected revenue from a first-price charity auction is

$$ER_1 = \int_{\underline{t}}^{\overline{t}} B_1(t)dF(t)^n. \tag{16}$$

The value of the integrand in (16) is always smaller when $\lambda > 0$ than it is when $\lambda = 0$, so expected revenue in a first price auction for $\lambda > 0$ is less than expected revenue from the same auction when $\lambda = 0$. Since a first-price auction with $\lambda = 0$ is revenue-equivalent to an all-pay auction with any value of λ, when $\lambda > 0$ a first-price auction raises less revenue than an all-pay auction.

We cannot apply a similar argument to a comparison of the expected revenue from a second price auction,

$$ER_2 = \int_{\underline{t}}^{\overline{t}} B_2(t)n(n-1)F(t)^{n-2}[1-F(t)]dF(t), \tag{17}$$

to the expected revenue of an all-pay auction,

$$ER_A = n\int_{\underline{t}}^{\overline{t}} B_A(t)dF(t). \tag{18}$$

The derivative of B_2 with respect to λ does not have the same useful properties as $\frac{\partial B_1}{\partial \lambda}$. In fact, no approach to comparing expected revenue from all-pay and second-price auctions will demonstrate that one selling format is always more lucrative than the other for all $n \geq 2$. We illustrate this point with the example of $t \sim U[0,1]$. When t is distributed uniformly on the unit interval, ER_A is larger than ER_2 when $n \geq 3$ but the revenue ranking depends on parameter values for $n = 2$.

Example 2 *Suppose t is distributed uniformly on $[0,1]$. Then the bid functions in second-price and all-pay auctions are*

$$B_2 = \left(\frac{1}{1-\Delta}\right)\left(\frac{1+\beta t}{1+\beta}\right) \qquad \textit{and} \qquad B_A = \left(\frac{1}{1-\theta}\right)\left(\frac{n-1}{n}\right)t^n.$$

Expected revenue from the two auction formats are

$$ER_2(n) = \frac{\lambda(n+1)+(1-\Delta)(n-1)}{(1-\Delta)(1-\Delta+\lambda)(n+1)} \qquad \textit{and} \qquad ER_A(n) = \left(\frac{1}{1-\theta}\right)\left(\frac{n-1}{n+1}\right).$$

The difference $ER_A(n) - ER_2(n)$ is proportional to

$$D(n) = \lambda[(n-3)(1-\theta) + 2\lambda(n-1)].$$

Clearly, if $n \geq 3$ then $D(n) > 0$ and $ER_A(n) > ER_2(n)$. If $n = 2$ then $ER_A(n) > ER_2(n)$ only if $(2\lambda + \theta) > 1$.

The intuition behind this revenue result is easier to grasp if we note that the all-pay charity auction is revenue-equivalent to a standard second-price auction in which bidders receive an *ad valorem* subsidy of $(1-\theta)^{-1}$. In this auction, bidders submit tenders equal to their "subsidized" valuations: $t/(1-\theta)$. Bidders in second-price charity auctions also offer their subsidized valuations, $t/(1-\Delta)$, plus an amount that is positive for all $t < \overline{t}$ and zero for $t = \overline{t}$. Equation (9), our initial expression for B_2, can be written simply as

$$B_2(t) = \frac{t}{1-\Delta} + \rho(t), \quad \text{with } \rho(t) = \left(\frac{1}{1-\Delta}\right)\int_t^{\overline{t}}\left[\frac{1-F(x)}{1-F(t)}\right]^{\beta} dx.$$

In the example above with $t \sim U[0,1]$, the function ρ is decreasing in t. As the number of bidders grows, the expected valuation of the second-highest bidder increases too, and the winner's payment is likely to be decided by a bidder with a low value of ρ. When the winner's payment is largely determined by the subsidized valuations, the all-pay auction is more lucrative than the second-price auction because $(1-\theta)^{-1} > (1-\Delta)^{-1}$ when $\lambda > 0$. This ranking of expected revenues may be reversed when the number of bidders is low, and the winner's payment is likely to be determined by a bidder with a relatively high value of $\rho(t)$.

Unfortunately, this reasoning does not hold for the general distribution function F. The function ρ is always decreasing in t for valuations sufficiently close to $\bar{t}$, but we cannot rule out that $\rho' > 0$ for some values of t. This implies that there may be some distribution functions for which $ER_A(n) > ER_2(n)$ but $ER_A(n+1) < ER_2(n+1)$. In the following proposition we describe expected revenue from charity auctions when n goes to ∞. In this limit ρ has no effect on the winner's payment.

Proposition 2. *The expected revenue from first-price, second-price, and all-pay auctions have the following properties:*

1. *First-price:* $\lim_{n\to\infty} [ER_1(n)] = \frac{\bar{t}}{1-\Delta}$,
2. *Second-price:* $\lim_{n\to\infty} [ER_2(n)] = \frac{\bar{t}}{1-\Delta}$,
3. *All-pay:* $\lim_{n\to\infty} [ER_A(n)] = \frac{\bar{t}}{1-\theta}$.

This proposition implies that, for $\lambda > 0$ and sufficiently large values of n, expected revenue from an all-pay auction is larger than that of either single-price format. Moreover, expected revenues from first- and second-price charity auctions converge as $n \to \infty$. The limiting values in Proposition 2 are also the winners' bids in these auctions. The winning bidder in an all-pay auction submits a bid that is high enough to leave her with zero additional surplus from winning (rather than withdrawing from the auction). Her benefit from other bidders' payments is unaffected by her own bid, so she is willing to compete away all additional surplus from winning. In the single-price auctions the bidder with the highest valuation must be permitted to retain some surplus from winning and making the only payment to the auctioneer. If not, this person could submit a bid of zero and receive a strictly positive benefit from allowing another person (of virtually the same valuation) to win the auction.

The relatively high revenue from an all-pay auctions with a large number of bidders invites comparison of this mechanism with other types of contests in which many people pay a small fee in return for a chance to win a prize. Consider a lottery or raffle in which proceeds of the contest are spent on a public good, as in Morgan [2000]. We conjecture that an all-pay auction would raise more revenue than a lottery because the auction always awards the prize to the person with the largest valuation. The auction is an efficient allocation mechanism, whereas a lottery is not. A person with a strong taste for a prize may submit a higher bid in an all-pay auction than she would be willing to spend on lottery tickets.

4.3 First-Price v. Second-Price Auctions

In order to establish this section's main result, we begin by stating a useful relationship between expected payments and expected revenue. Consider the expected payment of a person with the highest possible value, $\bar{t}$, for the auctioned object. This bidder is certain to win the auction in equilibrium, and in a first-price contest with n participants her payment is simply her bid:

$$P_1(\bar{t}, n) = \frac{1}{1-\Delta}\left\{\bar{t} - \int_{\underline{t}}^{\bar{t}} F(x)^{\alpha} dx\right\}.$$

While the payment of a bidder with type $\bar{t}$ in a first-price auction is known with certainty, this person's payment in a second-price auction is determined by the unobserved tastes of the bidder with the second-highest value of t.[12] The expected payment is:

$$P_2(\bar{t}, n) = \frac{1}{1-\Delta}\int_{\underline{t}}^{\bar{t}}\left\{t + \int_{t}^{\bar{t}}\left[\frac{1-F(x)}{1-F(t)}\right]^{\beta} dx\right\} dF(t)^{n-1}.$$

These payment terms may be inserted into the expected utility for a bidder of taste $\bar{t}$:

$$U_j(\bar{t}, n) = \bar{t} - (1-\theta)P_j(\bar{t}, n),$$

where the j subscript indicates (when necessary) the auction format.[13] The expected utility of a bidder with an arbitrary type, t, is:

$$U(t, n) = U(\bar{t}, n) - \int_{t}^{\bar{t}} \frac{\partial U(x, n)}{\partial x} dx.$$

By integrating U over values of t we obtain an average or *ex ante* expected utility to auction participants. This term is

$$E[U(t, n)] = U(\bar{t}, n) - \int_{\underline{t}}^{\bar{t}}\int_{t}^{\bar{t}} \frac{\partial U(x, n)}{\partial x} dx dF(t),$$

[12] It may be more helpful to think of the bidder who determines the winner's payment as the person with highest value of t among the remaining $(n-1)$ participants rather than the person with the second-highest of n values of t.

[13] We include the j subscript only in expressions that compare utility, expected payment, or expected revenue across auction formats.

and note that the difference in *ex ante* expected utility from the single-price auction formats depends only on the difference in P:

$$E[U_2(t,n)] - E[U_1(t,n)] = (1-\theta)[P_1(\bar{t},n) - P_2(\bar{t},n)]. \tag{19}$$

With this expression we have tied properties of utility for all bidders to a relationship between expected payments made by one particular type of bidder.

Now consider a second approach to expected utilities. Conditional on a bidder of type t winning a charity auction, the total expected utility to all bidders is:

$$\sum\nolimits_i [U(s_i,n) \mid t = \max\{s_j\}_{j=1}^n] = t - (1-\theta)P(t,n) + (n-1)\lambda P(t,n).$$

This expression is the value to the winner of receiving the object, t; the net utility from the winner's payment and her own benefit from making a transfer to the charity, $-(1-\theta)P$; and the benefit to all other auction participants from observing the winner's payment, $(n-1)\lambda P$. Integrating with respect to the winner's type, t, yields

$$\begin{aligned} E\left(\sum\nolimits_i [U(s_i,n)]\right) &= \int_{\underline{t}}^{\bar{t}} t dF(t)^n + [(n-1)\lambda - 1 + \theta] \int_{\underline{t}}^{\bar{t}} P(t,n) dF(t)^n \\ &= nE[U(t,n)]. \end{aligned} \tag{20}$$

The last integral in the expression above, the expected payment of the bidder with the highest type, is the expected revenue from the contest.[14] Thus, the difference in *ex ante* expected utilities from the two single-price auction formats is

$$E[U_2(t,n)] - E[U_1(t,n)] = \frac{1}{n}[(n-1)\lambda - 1 + \theta][ER_2(n) - ER_1(n)]. \tag{21}$$

We now have a pair of expressions, (19) and (21), that define a relationship between expected revenue and expected payment. Note that the differences $ER_2(n) - ER_1(n)$ and $P_1(\bar{t},n) - P_2(\bar{t},n)$ have the same sign if $(n-1) > \frac{1-\theta}{\lambda}$, and the differences have the opposite sign if the inequality is reversed. We establish our revenue ranking indirectly, by examining $P_1 - P_2$ and noting its implications for expected revenue.

Proposition 3. *$P_1 - P_2$ is positive when $(n-1) > \frac{1-\theta}{\lambda}$ and negative when $(n-1) < \frac{1-\theta}{\lambda}$. This implies that $ER_2 > ER_1$ for all admissible combinations of parameter values such that $(n-1) \neq$*

[14] The relationship between *ex ante* utility and expected revenue in equation (20) implies that bidders' welfare is increasing in auction revenue when $[(n-1)\lambda - 1 + \theta] > 0$. Prior to the realization of their valuations for an auctioned object, potential bidders may prefer that the auctioneer selects a revenue-maximizing auction format.

$\frac{1-\theta}{\lambda}$. *In the borderline case of* $(n-1) = \frac{1-\theta}{\lambda}$*, direct examination of expected revenue expressions reveals that* $ER_2 > ER_1$ *obtains again.*

In light of the revenue equivalence result for $\lambda = 0$, our finding in Proposition 3 must be attributed to the benefit from other players' payments on B_1 and B_2. Given that a particular bidder will lose the auction, there is an incentive (and ability) to increase the winner's payments in a second-price auction that is not present in a first-price contest. This result on expected revenue accords with common practice in charity auctions. A fairly popular format for charity auctions of multiple objects is to sell the less valuable items in a "silent auction" while guests mingle, and the items with higher expected prices are sold in oral, ascending-bid auctions.

5 Conclusions

In this paper we have studied equilibrium bidding strategies in charity auctions. Three selling formats were examined: all-pay auctions and first- and second-price auctions. We found that participants in all types of auctions bid more aggressively when they benefit from revenue collected by the host charity. If bidders receive no benefit from the payments made by bidders other than themselves, they simply inflate the tenders that they would have made in a standard (non-charity) auction by a constant. Revenue equivalence among the auction formats is preserved in this case. Bidding incentives are generally more complicated when auction revenue is like a public good that all bidders enjoy regardless of which bidder(s) contributed revenue. An increase in the benefit from others' payments can depress bids in the single-price auctions, and the auctions each have different expected revenue. All-pay auctions are the most lucrative format when the number of bidders is sufficiently large, and second-price auctions have higher expected revenue than first-price auctions.

In practice charity auctions can be quite complicated. Many fund-raising events include the sale of several objects in a sequence of auctions. Minimum bids and entrance fees for charity events are also typical (and important to overall revenue). For example, in 2000 the Napa Valley Wine Auction had an entrance fee of $1,000. We intend to study these issues in future research. Given the similarity of equilibrium bidding in an all-pay charity auction to that in the non-charity case, we suspect that the optimal entrance fee or minimum bid for an all-pay auction will not be substantially different from those in the standard setting. However, these rules and fees in first- and second-price auctions are likely to be different from those in non-charity auctions and different from each other. A broader set of research questions exists for considering when a charitable organization would prefer to hold an auction instead of using a different fund-raising mechanism. It may be the case that auctions are most useful when potential contributors are particularly unwilling to open their wallets for a charity. We have shown that charitable groups can use auctions to attract revenue (above what would be collected in a non-charity auction) from potential contributors who otherwise would not be eager to make voluntary donations to the organization.

A Appendix

We begin by establishing that the we have derived equilibrium bidding functions for the three auction formats considered in this paper. Following this, we re-state and prove the propositions offered above.

A.1 Equilibrium bid functions

The bid functions we presented in Section 3 constitute equilibrium bidding rules if they satisfy the initial assumptions we stated in Section 2 (*i.e.*, they are increasing and differentiable) and they are optimal solutions to individual bidders' signalling problems. We consider each auction format in turn to verify that these conditions hold.

First-price auctions

B_1 **is increasing and differentiable.** The derivative of B_1,

$$B_1'(t) = \left(\frac{\alpha}{1-\Delta}\right)\frac{f(t)}{F(t)}\int_{\underline{t}}^{t}\left(\frac{F(x)}{F(t)}\right)^{\alpha} dx,$$

exists for all $t \in (\underline{t}, \overline{t})$ and is strictly positive.

B_1 **is optimal.** The following proof applies to the other types of auction as well. For each type of auction we can write $\pi(s|t,n) = (t-s)F(s)^{n-1} + \pi(s|s,n) = (t-s)F(s)^{n-1} + U(s)$, where we let $U(s)$ denote $\pi(s|s,n)$. As long as the first-order condition for a bidder of type t is satisfied we can apply the envelope theorem to obtain $U'(t) = F(t)^{n-1}$, which establishes that the function U is increasing and strictly convex because its derivative, $F(t)^{n-1}$, is positive and strictly increasing in t. We show that, for any distinct s and t in $[\underline{t}, \overline{t}]$ we have $\pi(s|t,n) < \pi(t|t,n)$, which can be rewritten as $(t-s)F(s)^{n-1} + U(s) < U(t)$, or equivalently, for $s \in (\underline{t}, \overline{t})$, $(t-s)U'(s) + U(s) < U(t)$.

This last inequality follows immediately from the fact that U is strictly convex. For $s = \underline{t}$, the required inequality is equivalent to $U(\underline{t}) < U(t)$ which is true because U is increasing. Similarly, for $s = \bar{t}$, the inequality is equivalent to $t - \bar{t} + U(\bar{t}) < U(t)$ which is true because $U'(t) < 1$ for $t < \bar{t}$.

Second-price auctions

B_2 **is increasing and differentiable.** The derivative of B_2,

$$B_2'(t) = \left(\frac{\beta}{1-\Delta}\right)\left(\frac{f(t)}{1-F(t)}\right)\int_t^{\bar{t}}\left(\frac{1-F(x)}{1-F(t)}\right)^{\beta} dx,$$

exists for all $t \in (\underline{t}, \bar{t})$ and is strictly positive.

All-pay auctions

B_A **is increasing and differentiable.** The derivative of B_A,

$$B_A'(t) = \frac{t}{1-\theta}\frac{dF(t)^{n-1}}{dt},$$

exists for all $t \in (\underline{t}, \bar{t})$ and is strictly positive.

A.2 Propositions and proofs

Proposition 1. *The following is a Perfect Bayesian Equilibrium in a button auction: Each bidder of type t who has not yet released her button will release her button at the announced price p if and only if $p \geq B_2(t)$. The outcome of the button auction in terms of allocations and payments is identical to the second-price sealed-bid auction. Moreover, these results for the button auction hold whether or not bidders observe when their rivals exit.*

Proof. We begin with a simplified static game which operates in the same way as the button auction except that all players choose simultaneously and once-and-for-all when their buttons are to be released. It should be clear from the rules of the button auction that this static game has exactly the same payoff structure as the second-price sealed-bid auction, and hence it is a Bayesian Nash equilibrium for each player of type t to release her button at the price $B_2(t)$.

Next consider the case in which bidders can decide whether to exit as the price indicator rises, and bidders can observe when rivals exit. Assume that all bidders other than i are using B_2 to determine their exit prices, and consider player i's optimal response to this situation as the price indicator rises. Let i have type t. At any price p person i observes the number $m \leq n$ of bidders that are still "live" and can conclude that they are of type $s \geq x$, where $x = B_2^{-1}(p)$. We argue that under these conditions if $p \leq B_2(t)$ then the bidder of type t will find it optimal to wait until the price is $B_2(t)$ to exit. The reason is that the expected payoffs as a function of exit price b are exactly the same as those in a second-price auction with bid b, in which there are m bidders and the rivals each have types drawn independently from the distribution $F(s)$ conditional on $s \geq x$.

Examination of the function B_2 shows that it is invariant with respect to $m \leq n$, the number of bidders who are active in the auction. B_2 is also unaltered when the unconditional distribution $F(s)$ is replaced by the appropriate conditional distribution $\frac{F(s)-F(x)}{1-F(x)}$. This implies that at any price p in the button auction it is optimal for i to wait until $B_2(t)$ if $p < B_2(t)$, and to exit immediately once $p \geq B_2(t)$. Thus the posited strategies constitute a Perfect Bayesian Equilibrium.

Finally consider the situation in which bidders do not observe others' exit decisions. Now all that a live bidder knows is that there is at least one other live bidder left (or the auction would have ended). The previous result showed that the posited strategies are optimal regardless of how many other bidders remain, and so, in this case too, the strategies comprise a Perfect Bayesian Equilibrium. **Q.E.D.**

Proposition 2. *The expected revenue from first-price, second-price, and all-pay auctions have the following properties:*

1. *First-price:* $\lim_{n\to\infty} [ER_1(n)] = \frac{\bar{t}}{1-\Delta}$,
2. *Second-price:* $\lim_{n\to\infty} [ER_2(n)] = \frac{\bar{t}}{1-\Delta}$,
3. *All-pay:* $\lim_{n\to\infty} [ER_A(n)] = \frac{\bar{t}}{1-\theta}$.

Proof. We prove this proposition by establishing each listed result separately.

1. **First-price.** Consider how the function B_1 changes with n. As $n \to \infty$ the exponent, α, in the integrand of B_1 also goes to infinity. Since $\frac{F(x)}{F(t)}$ is always less than one for $x \in (\underline{t}, t)$, the integral
$$\int_{\underline{t}}^{t} \left(\frac{F(x)}{F(t)}\right)^{\alpha} dx$$
disappears as $n \to \infty$. $B_1(t)$ approaches $\frac{t}{1-\Delta}$. Now consider the valuation of the bidder that wins the auction. The distribution of the highest draw from F becomes concentrated at $\bar{t}$

as $n \to \infty$, so the only bid value in (16) that is given any weight is $B_1(\bar{t}) = \frac{\bar{t}}{1-\Delta}$. Therefore ER_1 approaches $\frac{\bar{t}}{1-\Delta}$ as $n \to \infty$.

2. **Second-price.** B_2 is independent of n, but as $n \to \infty$ the distribution of the second-highest draw from F becomes concentrated at $\bar{t}$. The bid of an individual with type $\bar{t}$ is the only bid given any weight in the expected revenue from a second price auction, (17). Since $B_2(\bar{t}) = \frac{\bar{t}}{1-\Delta}$, ER_2 approaches $\frac{\bar{t}}{1-\Delta}$ as $n \to \infty$.

3. **All-pay.** In Section 4 we argued that the expected revenue from an all-pay charity auction is $(1-\theta)^{-1}$ times the revenue from a standard all-pay (or first- or second-price) auction. Because of the revenue equivalence result for standard auctions, the expected revenue from a standard auction is the expected value of the second-highest draw from the type distribution F. As the number of bidders approaches ∞, the expected value of the second-highest valuation approaches $\bar{t}$, as does the expected revenue in a standard auction. Therefore, as $n \to \infty$, ER_A goes to $\frac{\bar{t}}{1-\theta}$. **Q.E.D.**

Proposition 3. *$P_1 - P_2$ is positive when $(n-1) > \frac{1-\theta}{\lambda}$ and negative when $(n-1) < \frac{1-\theta}{\lambda}$. This implies that $ER_2 > ER_1$ for all admissible combinations of parameter values such that $(n-1) \neq \frac{1-\theta}{\lambda}$. In the borderline case of $(n-1) = \frac{1-\theta}{\lambda}$, direct examination of expected revenue expressions reveals $ER_2 > ER_1$ obtains again.*

The following lemma is necessary for the proof of Proposition 3 but holds no economic content, so it is presented in this appendix only. The proof of Proposition 3 follows.

Lemma 1. Suppose that $z : [a,b] \to \Re$ is continuous on $[a,b]$, differentiable on (a,b), and $z = 0$ at a and b. If z is positive wherever its derivative vanishes, then z is positive on (a,b). Alternatively, if z is negative wherever $z' = 0$, then z is negative on (a,b).

Proof of Lemma 1. We prove the first part of the lemma (which concludes z is positive on (a,b)) by proving the contrapositive. Suppose $z(x) \leq 0$ for some x in (a,b). Then either x is a global minimum or there is a global minimum y such that $z(y) < z(x) \leq 0$. In either case, we have found a global minimum in (a,b). As this is an interior local minimum, z' must vanish at a point where z is not positive. A similar argument proves the second part of the lemma. **Q.E.D.**

Proof of Proposition 3. In Case 1 of this proof we suppose that $(n-1) \neq \frac{1-\theta}{\lambda}$, and the revenue result is established by examination of $P_1 - P_2$. In Case 2 we assume $(n-1) = \frac{1-\theta}{\lambda}$ and prove $ER_2 > ER_1$ directly.

Case 1:

Our first step is to rewrite $P_2(\bar{t}, n)$ as

$$\begin{aligned} P_2(\bar{t}, n) &= \frac{1}{1-\Delta}\left\{\int_{\underline{t}}^{\bar{t}} t dF(t)^{n-1} + \int_{\underline{t}}^{\bar{t}}\int_{t}^{\bar{t}}\left[\frac{1-F(x)}{1-F(t)}\right]^{\beta} dx dF(t)^{n-1}\right\} \\ &= \frac{1}{1-\Delta}\left\{\bar{t} - \int_{\underline{t}}^{\bar{t}} F(t)^{n-1} dt + \int_{\underline{t}}^{\bar{t}}\int_{\underline{t}}^{x} \frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}}[1-F(x)]^{\beta} dx\right\}. \end{aligned}$$

Now the difference in expected prices depends on the value of an expression integrated from $\underline{t}$ to $\bar{t}$:

$$P_1(\bar{t}, n) - P_2(\bar{t}, n) = \frac{1}{1-\Delta}\int_{\underline{t}}^{\bar{t}}\left\{F(x)^{n-1} - F(x)^{\alpha} - \left(\int_{\underline{t}}^{x}\frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}}\right)[1-F(x)]^{\beta}\right\} dx.$$

Let $g(x)$ represent the terms in braces, so that

$$P_1(\bar{t}, n) - P_2(\bar{t}, n) = \frac{1}{1-\Delta}\int_{\underline{t}}^{\bar{t}} g(x) dx.$$

We verify the sign of $P_1 - P_2$ by showing that the integrand g is either positive $\forall$ x or negative $\forall$ x, depending on parameter values and the number of bidders.

We now demonstrate that Lemma 1 applies to g. The continuity and differentiability of g follow from the assumption that F has these properties; substitution of $\underline{t}$ and $\bar{t}$ into g sets the value of this function to zero. Differentiation yields

$$\begin{aligned} g'(x) &= (n-1)F(x)^{n-2}f(x) - \alpha F(x)^{\alpha-1}f(x) - \frac{(n-1)F(x)^{n-2}f(x)}{[1-F(x)]^{\beta}}[1-F(x)]^{\beta} \\ &\quad + \left(\int_{\underline{t}}^{x}\frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}}\right)\beta[1-F(x)]^{\beta-1}f(x) \\ &= \left(\int_{\underline{t}}^{x}\frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}}\right)\beta[1-F(x)]^{\beta-1}f(x) - \alpha F(x)^{\alpha-1}f(x). \end{aligned}$$

And since the density is positive

$$g'(x) = 0 \Leftrightarrow \left(\int_{\underline{t}}^{x}\frac{dF(t)^{n-1}}{[1-F(t)]^{\beta}}\right) = \frac{\alpha F(x)^{\alpha-1}}{\beta[1-F(x)]^{\beta-1}}.$$

Consider an $x \in (\underline{t}, \bar{t})$ at which $g'(x) = 0$. We now show that at any such x, $g(x) > 0$. When $g'(x) = 0$ it must be true that

$$g(x) = F(x)^{n-1} - F(x)^{\alpha} - \frac{\alpha}{\beta}F(x)^{\alpha-1}[1-F(x)].$$

The sign of $g(x)$ is unaffected by division by $F(x)^{n-1}$, which yields

$$h(x) = 1 - [1-q]F(x)^q - qF(x)^{q-1},$$

where $q = \lambda(n-1)/(1-\theta)$. The new parameter q takes values in $(0, \infty)$, and the relevant cases for this parameter are when $q > 1$ ($P_1 - P_2$ and $ER_2 - ER_1$ have the same sign) and $q < 1$ ($P_1 - P_2$ and $ER_2 - ER_1$ have the opposite sign). The case of $q = 1$ is addressed below.

Our approach is to show that the sign of h is positive (negative) for all $x \in (\underline{t}, \bar{t})$ when $q > 1$ ($q < 1$). If h has the appropriate sign for all x, then g will have the same sign as h whenever $g' = 0$. First note that when $q > 1$, it is the true that $\lim_{x \to \underline{t}} h(x) = 1$, $\lim_{x \to \bar{t}} h(x) = 0$, and $h'(x) < 0 \; \forall \; x \in (\underline{t}, \bar{t})$. Together, these conditions imply that $h(x) > 0 \; \forall \; x \in (\underline{t}, \bar{t})$ when $q > 1$, so we may also conclude that $g > 0$ when $g' = 0$. By Lemma 1 we know that this implies $g > 0$ for all $x \in (\underline{t}, \bar{t})$, so $P_1 > P_2$. In Section 4.3 we established that $P_1 > P_2 \Rightarrow ER_2 > ER_1$ when $q > 1$. Now consider $q < 1$, for which $\lim_{x \to \underline{t}} h(x) = -\infty$, $\lim_{x \to \bar{t}} h(x) = 0$, and $h'(x) > 0 \; \forall \; x \in (\underline{t}, \bar{t})$. These conditions imply $h(x) < 0 \; \forall \; x \in (\underline{t}, \bar{t})$ when $q < 1$, so we may also conclude that $g < 0$ when $g' = 0$. By Lemma 1 we know that this implies $g < 0$ for all $x \in (\underline{t}, \bar{t})$, so $P_1 < P_2$. In Section 4.3 we established that $P_1 < P_2 \Rightarrow ER_2 > ER_1$ when $q < 1$.

Case 2:

The maintained assumption of $(n-1) = \frac{1-\theta}{\lambda}$ implies that the parameters α and β, which appear in B_1 and B_2, are both equal to n. This simplifies expected revenue terms and facilitates their direct comparison. These terms are:

$$\begin{aligned}
ER_1 &= \frac{1}{1-\Delta} \int_{\underline{t}}^{\bar{t}} \left\{ t - \int_{\underline{t}}^{t} \left[\frac{F(x)}{F(t)} \right]^n dx \right\} dF(t)^n \\
&= \frac{1}{1-\Delta} \left\{ \int_{\underline{t}}^{\bar{t}} t dF(t)^n - \int_{\underline{t}}^{\bar{t}} \int_{x}^{\bar{t}} \frac{dF(t)^n}{F(t)^n} F(x)^n dx \right\} \\
&= \frac{1}{1-\Delta} \left\{ \bar{t} - \int_{\underline{t}}^{\bar{t}} F(t)^n dt + n \int_{\underline{t}}^{\bar{t}} [\log F(t)] \, F(t)^n dt \right\} \\
ER_2 &= \frac{1}{1-\Delta} \int_{\underline{t}}^{\bar{t}} \left\{ t + \int_{t}^{\bar{t}} \left[\frac{1-F(x)}{1-F(t)} \right]^n dx \right\} dF_{(2)}^n(t) \\
&= \frac{1}{1-\Delta} \left\{ \int_{\underline{t}}^{\bar{t}} t dF_{(2)}^n(t) + \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n} [1-F(x)]^n dx \right\} \\
&= \frac{1}{1-\Delta} \left\{ \bar{t} - \int_{\underline{t}}^{\bar{t}} F_{(2)}^n(t) dt + \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n} [1-F(x)]^n dx \right\},
\end{aligned}$$

with $F_{(2)}^n(t)$ as the distribution function of the second-highest of n draws from F. Next, define the function $g(x)$ so that

$$ER_2 - ER_1 = \frac{1}{1-\Delta}\int_{\underline{t}}^{\bar{t}} g(x)dx.$$

From the expected revenue expressions above we know that

$$g(x) = \int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n}[1-F(x)]^n - nF(x)^{n-1}[1-F(x)] - n\left[\log F(x)\right]F(x)^n.$$

Differentiation of g yields

$$\begin{aligned} g'(x) &= \frac{n(n-1)F^{n-2}(x)[1-F(x)]f(x)}{[1-F(x)]^n}[1-F(x)]^n - \\ &\quad \int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n} n[1-F(x)]^{n-1}f(x) - n(n-1)F(x)^{n-2}[1-F(x)]f(x) + \\ &\quad nF(x)^n f(x) - nF(x)^{n-1}f(x) - n^2\left[\log F(x)\right]F(x)^{n-1}f(x) \\ &= -\int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n} n[1-F(x)]^{n-1}f(x) - n^2\left[\log F(x)\right]F(x)^{n-1}f(x). \end{aligned}$$

This derivative is equal to zero when

$$\int_{\underline{t}}^{x} \frac{dF_{(2)}^n(t)}{[1-F(t)]^n} = -\frac{n\left[\log F(x)\right]F(x)^{n-1}}{[1-F(x)]^{n-1}},$$

Consider an $x \in (\underline{t}, \bar{t})$ at which $g'(x) = 0$. We now show that at any such x, $g(x) > 0$. When $g'(x) = 0$ it must be true that

$$g(x) = -n\left[\log F(x)\right]F(x)^{n-1} - nF(x)^{n-1}[1-F(x)].$$

The sign of this expression is unaffected by dividing by $nF(x)^{n-1}$, so we do this to obtain the simpler expression

$$h(x) = -\log[F(x)] - [1-F(x)].$$

As in Case 1, we show that the sign of h is constant for all x in order to establish the sign of g whenever $g' = 0$. The function h has the following properties: $\lim_{x\to\underline{t}} h(x) = \infty$, $\lim_{x\to\bar{t}} h(x) = 0$, and $h'(x) < 0 \ \forall \ x \in (\underline{t}, \bar{t})$. Together, these conditions imply $h(x) > 0 \ \forall \ x \in (\underline{t}, \bar{t})$, so we may also conclude that $g > 0$ when $g' = 0$. By Lemma 1, we know that this implies that $g(x) > 0 \ \forall \ x \in (\underline{t}, \bar{t})$, so it must be the case that $ER_2 > ER_1$ when $(n-1) = \frac{1-\theta}{\lambda}$. **Q.E.D.**

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FIGURE 1