

Search Direction*

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Abstract

A tractable model of pricing under directed search is proposed and integrated with a position auction for better slots (which rationalizes the consumer search order). The equilibrium search order resulting from the auction may be socially optimal or not, depending on the nature of product heterogeneity. Search is always inefficiently low because firms price out further exploration. Equilibrium product prices are such that the marginal consumer's surplus decreases in the order of search. Consumers always find it optimal to follow the order of search that results from the auction. Equilibrium bids factor in position externalities across firms as prices and profits depend on the qualities of firms following in the sequence of positions. We highlight the fundamental role of firm heterogeneity that characterizes markets and their performance. The search framework delivers a full-fledged integration of position auctions and pricing with sequential directed search.

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1 Introduction

Internet search is sequential across options, and surfer search is directed by position placement of ads determined by advertisers paying for earlier positions. Until now, most work on consumer search and firm pricing has been with random search across options (e.g. the work following Stahl's mixed strategy model with homogenous goods, or the search for match following Wolinsky, 1986, and Anderson-Renault, 1999). Directed search is quite different, and the theory needs to be developed. Research has been stymied so far by lack of tractable frameworks that can accommodate heterogeneous firms, a key ingredient for the analysis of directed search. We propose a tractable framework that reflects the internet search environment (and most other contexts too!) and engages Weitzman's (1979) powerful results on search behavior. It enables us to study equilibrium and optimal consumer search, product pricing, and advertiser bidding for positions. It delivers a falling surplus for the marginal consumer in the order of search, with consumers (strictly) wanting to follow the order directed from the position bidding auction. Pricing excessively curtails search. The socially optimal order, joint profit maximizing order and consumer surplus maximizing order may each be characterized by associating a score to each firm and ranking the firms according to that score. Equilibrium bids reflect positional externalities; the search order is typically not socially optimal.

An important property of our model is the externality imposed by a firm's position on other firms' profits. This effect is only partially incorporated if at all in the literature on position auctions. In our context, this also means that a firm's willingness-to-pay for a slot depends on which firm is demoted and the distribution of tastes for its product. To see these effects, note that if a firm in the i th position becomes more attractive to search, then the prices and profits of firms before it are reduced. Thus the willingness to pay for a slot depends on who is where.

Consider now a position auction for slots, with slots going to firms in the order of their bids, and firms paying the bid of the next highest bidder. Such auctions generally admit multiple equilibria. Following Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007),

we therefore consider a “no-envy” refinement. This means that no firm would like the position of another if it had to pay the price the other is paying for its slot. When a firm bids to up its slot, it recognizes that it demotes others and thus changes its equilibrium price. The important result is that we can show (under this refinement) that there is a unique order of firms induced by the auction. This order usually does not coincide with the social optimum. For some specification of the model, consumers would prefer the reverse order.

One key to a broader understanding of the link between equilibrium search and positions is to look at asymmetries in the other variables of the model. Fortunately, it is populated with several parameters that play differently and can be distributed across firms. These we break out in the model.

There are two relevant streams of literature. Sequential ordered search has only recently been broached. A major step forward was made by Armstrong, Vickers, and Zhou (2009), who showed that a firm which is searched first will earn more profit, but will also be more attractive to consumers to search first. However, their model has a single “prominent” firm searched first, and then the remaining firms are searched at random (without order). Moreover, they assume a uniform distribution of consumer tastes to establish their case, which limits introducing heterogeneity into the distribution of tastes. Zhou (2011) addresses some of these concerns with an ordered search model, again with a uniform distribution and symmetric firms. Song (2012) still using a uniform distribution considers firms that are asymmetric regarding taste heterogeneity, but only looks at the duopoly case. Finally, Chen and He (2011) introduce some heterogeneity in the probability that a product is suitable for a consumer, although their model delivers monopoly pricing hence there is no externality through prices.

The position auctions literature has made valuable progress on the auction side of the slate while suppressing the market competition side. Athey and Ellison (2011) use a setting very similar to that of Chen and He (2011) to look at auctions with asymmetric information and then optimal auction design, while assuming that consumers go on searching until a “need” is fulfilled, so they do not allow for competing products on the market-place. Their

setting allows for a position externality through demand which depends on how likely are previous products in the queue to fill a consumer’s need. Our setting does allow for this type of externality as well as the pricing externalities described above. Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007) have a sparse market competition description without position externalities between firms, and they do not engage the broader consumer search and pricing either.

These papers analyze per click auctions, with a price paid per click. Alternatives are auctions per conversion (a successful click) or per position (without conditioning on how many clicks are made on the ad). Athey and Ellison (2009) further analyze a situation where bidders and consumers know only the distribution of parameters from which any particular bidder type is drawn. By contrast, we analyze the situation where payoffs are allowed to depend on the identities of who is before or after. We also are interested from the search theory perspective in the implications for product pricing, which is suppressed in the above settings

2 Market equilibrium

2.1 Competition with ordered search

We first describe a basic model of oligopolistic competition with ordered search, where the order of search is exogenous.

Consumers have independent valuations for n competing products. The valuations for product i , $i = 1, \dots, n$, are either 0, $q_i > 0$, or $q_i + \Delta_i$ where $\Delta_i > 0$,¹ and q_i is taken as “sufficiently large”, as explained below. Let the corresponding match probabilities be $\gamma_i = 1 - (\alpha_i + \beta_i)$, α_i , and β_i . This structure begets a two-step demand function, and is the simplest in which we can get to the essence of ordered search.

Search is sequential and ordered, with the search cost $s > 0$ per additional search. As is standard in sequential search settings the consumer may always purchase from any previously

¹Alternatively, Δ_i is the expectation of the surplus increment over the base quality, conditional on it being strictly positive. In particular, if this positive increment has a continuous distribution with a logconcave density, the monopoly price is q_i , if q_i is large enough.

searched firm with no additional search cost. There are n firms with firm i selling product i , with zero production costs. The order of search is from the lowest to the highest value of i . We seek conditions for a particular pricing equilibrium, namely that each firm retains all consumers with a non zero match value. This means that we seek an equilibrium where firms render indifferent any consumer drawing q_i : the constraint therefore is that further search is not desirable for such a consumer. This implies that a consumer has zero willingness to pay for any product encountered before product i , and, as long as prices are strictly positive, never goes back.

Let V_i denote the minimum value a consumer must hold at firm $i = 1, \dots, n - 1$ to give up searching on. It is defined by $E_{v_i} \max\{v_i - V_i, 0\} = s$, where v_i is the realization of a random variable measuring the best surplus the consumer can obtain by searching optimally from firm $i + 1$ on and we use the free recall assumption. Hence firm i 's price must be such that

$$q_i - p_i = V_i. \tag{1}$$

We now derive the equilibrium prices.

2.2 Pricing

Consider first the pricing problem of the last firm in the queue, firm n . Since consumers who reach firm n have a zero valuation of products at previously visited firms, it behaves like a monopolist against some continuation value V_n . For now we treat V_n as exogenous, merely assuming that it is identical for all consumers and positive. Allowing for $V_n > 0$ means that once the consumer is done going through the n firms, she still has additional options to purchase a product. This could be for instance, searching the organic links of a search engine after searching the sponsored link, or purchasing a product off line. For q_n large enough, the firm will choose to price at $p_n = q_n - V_n$. Suppose then that a consumer at the firm in position $n - 1$ holding surplus $q_{n-1} - p_{n-1}$ contemplates searching firm n . Given she expects firm n 's optimal pricing behavior, and using (1) for $i = n - 1$, her continuation value from searching on may be written as

$$V_{n-1} = (\gamma_n + \alpha_n) \max\{V_{n-1}, V_n\} + \beta_n \max\{V_{n-1}, V_n + \Delta_n\} - s, \tag{2}$$

where we use the free recall assumption. If $V_n > V_{n-1}$, then we have $V_{n-1} = V_n + \omega_n$, where $\omega_n = \beta_n \Delta_n - s$. This is only possible if $\omega_n < 0$. If V_n is too small and ω_n is sufficiently negative so that $V_{n-1} < 0$, the consumer prefers dropping out rather than searching on to firm n . Then firm $n - 1$ can retain her while charging the monopoly price (which is q_{n-1} if q_{n-1} is large enough), so the Diamond paradox would apply. If $V_{n-1} \geq 0$ although $\omega_n < 0$, then the consumer had rather search beyond firm n directly and only searches firm n because of the option value it gives her to search beyond. This is a situation where the consumer is forced to search through options in a suboptimal order and the standard search theoretic results in Weitzman (1979) do not apply. In our analysis below, we allow for the consumer to deviate from the specified search order and we wish to consider firm pricing outside the Diamond paradox.² We therefore assume $\omega_i > 0$ for all i .

Hence we must have $V_{n-1} > V_n$ and from (2), $V_{n-1} = V_n + \frac{\omega_n}{\beta_n}$ (This is because, in order for (2) to hold with $s > 0$, we need $V_n + \Delta_n > V_{n-1}$). From (1) we have $p_{n-1} = q_{n-1} - \frac{\omega_n}{\beta_n} - V_n$. We now use a similar line of argument to establish by induction the following result.

Proposition 1 *If $\omega_i > 0$ and q_i is large enough, then there exists an equilibrium that satisfies*

$$V_i = V_n + \sum_{j=i+1}^n \frac{\omega_j}{\beta_j}, \quad (3)$$

for all $i = 1, \dots, n - 1$, so that

$$p_i = q_i - \sum_{j=i+1}^n \frac{\omega_j}{\beta_j} - V_n, \quad (4)$$

for all $i = 1, \dots, n$.

Proof. First, we have already established (3) for $i = n - 1$ and (4) for $i = n$. Now, because of (1), if (3) holds for $i = 1, \dots, n - 1$, then pricing satisfies (4) for $i = 1, \dots, n - 1$. Hence, to prove the result it suffices to show by induction that if (3) is true for some $i = 2, \dots, n - 1$ then it is true for $i - 1$.

²Previous literature has used settings where the Diamond paradox applies, as in Chen and He, 2013.

Consider a consumer at firm $i - 1$, holding surplus $q_{i-1} - p_{i-1}$. Since firm i 's price satisfies $q_i - p_i = V_i$, her expected surplus from searching may be written as

$$V_{i-1} = (\gamma_i + \alpha_i) \max\{V_{i-1}, V_i\} + \beta_i \max\{V_{i-1}, V_i + \Delta_i\} - s, \quad (5)$$

The arguments used to derive V_{n-1} can be replicated here to show that, $\omega_i > 0$ implies that $V_{i-1} \geq V_i$ and hence $V_{i-1} = V_i + \frac{\omega_i}{\beta_i}$ (again, in order for (5) to hold with $s > 0$, we must have $V_{i-1} < V_i + \Delta_i$). Thus if V_i satisfies (3), so does V_{i-1} . ■

2.3 Directed search.

Our equilibrium analysis thus far has assumed that consumers must search in a set order. We have established that, if $\omega_i > 0$ for all i and firms choose to retain all consumers with a positive valuation with their product, then equilibrium prices are given by (4) and such an equilibrium exists if q_i is large enough for all $i = 1, \dots, n$.³ This characterization of equilibrium does not require that the order of search is optimal for consumers. In particular, it does not rely on the standard myopic reservation value rule of Weitzman (1979). As has been already pointed out, we do want our characterization to be robust to the possibility that a consumer freely selects the order in which she searches. We now show that, for the equilibrium pricing rule derived above, the pre specified search order is always optimal, for all values of q_i and Δ_i , as long as $\omega_i > 0$ for all $i = 1, \dots, n$. Zhou (2011) also found that firm pricing makes it optimal for consumers to start searching the firms that expect to be searched early. In the symmetric case he considers, this follows immediately from the higher prices charged by firms later in the queue. We extend this result to asymmetric products. The underlying force here is that the marginal consumer's surplus is lower with firm that are searched later. Indeed, from equation (1) this surplus is V_i at firm i , and from equation (4) V_i is strictly decreasing in i .

As shown by Weitzman (1979), in order to determine the optimal search order, it suffices to compute a reservation value associated with each search alternative: it is then optimal to

³We expect that if q_i is large enough for all i , there cannot be any other equilibrium. Such alternative equilibrium would have some firms price in a way such that only consumers with the highest valuation stop searching when they get to the firm.

search the alternatives following the decreasing order of reservation values. in our setting, the consumer's utility with product i is $u_i = 0$ with probability γ_i , $u_i = q_i - p_i = V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$ with probability α_i and $u_i = q_i + \Delta_i - p_i = \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$ with probability β_i . Then the reservation utility associated with searching firm i , \hat{u}_i satisfies

$$E_{u_i} \max\{u_i - \hat{u}_i, 0\} = \gamma_i \max\{-\hat{u}_i, 0\} + \alpha_i \max\{V_n + \sum_{j>i} \frac{\omega_j}{\beta_j} - \hat{u}_i, 0\} + \beta_i \max\{\Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j} - \hat{u}_i, 0\} = s \quad (6)$$

The left-hand side is zero for $\hat{u}_i = \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$. It is continuous and strictly decreasing in \hat{u}_i . Hence, for $s > 0$ (6) has a unique solution $\hat{u}_i < \Delta_i + V_n + \sum_{j>i} \frac{\omega_j}{\beta_j}$. It is readily verified that $\hat{u}_i = V_n + \sum_{j \geq i} \frac{\omega_j}{\beta_j}$. This is clearly decreasing in i so it is optimal for the consumer to search earlier firms first. Note that $\hat{u}_i > V_n$ for all $i = 1, \dots, n$, so that the consumer finds it optimal to search any of the n firms rather than moving on directly to her best alternative shopping strategy (e.g. organic links or shopping off line). Furthermore, for $i = 2, \dots, n$, $\hat{u}_i = V_{i-1}$, which, from (1), is the marginal consumer's surplus at firm $i - 1$. This reflects firm $i - 1$'s strategy to make the consumer drawing q_{i-1} indifferent between buying and searching on.

Results thus far establish that for any order in which the firms are ranked, there is a pricing equilibrium such that each firm retains all consumers who reach it and are willing to pay some positive amount of money for its product, and consumers find it optimal to search according to the pre specified order. We next investigate on what basis this ranking could be determined.

3 Optimal rankings

Here we look at various ranking criteria of search orders. What is the best order for consumer surplus, social welfare and total industry profit ? Given asymmetries across firms in the parameters, which order of presentation (given equilibrium search and pricing) maximizes these? A priori, this is a complicated problem because position order affects all prices and

search probabilities: with n active firms there are $n!$ positions to check. Nevertheless, our model delivers a simple structure such that we can simply characterize the optimal order under each criterion, and the optimal order is described by ordering a simple summary statistic (different for each criterion).

The idea is as follows. Suppose that we rank firms some arbitrary way. Then for any neighboring pair of firms, A and B, in the ranking (and for each criterion), we can find a summary statistic Ω_k for firms such that the maximand (CS, W, or TIP) is higher if $\Omega_A > \Omega_B$. Crucially, whether or not $\Omega_A > \Omega_B$ does not depend on which two slots are flipped (e.g., first and second or fifteenth and sixteenth). Suppose for clarity (and to eliminate ties, which have no consequence anyway – the order is then indifferent between when tied firms are presented – that the Ω_k are all different across firms. Then the claimed result is that there is a unique maximum, and simply follows the order of the Ω_k . Clearly a necessary condition is that in each successive pair the one with the higher Ω_k goes first – otherwise we can increase the maximand by flipping any pair which violates this. But then, because the flipping rule is independent of the positions i and $i + 1$ to be flipped, this criterion just promotes up the order each firm to the positions claimed. Put another way, for any order not satisfying the claimed optimal ranking, there must be at least one pair violating the pairwise flip condition, and so this cannot be an optimum.

We now derive the particular summary statistics for the different criteria. We also look at some intuition for the various orders.

3.1 TIP

For TIP, we just need to look at the change in (gross) profit from the switch. Thus we have A before B as long as

$$\pi_A^i + \pi_B^{i+1} \geq \pi_B^i + \pi_A^{i+1} \tag{7}$$

For our model, we can write this out to yield:

$$(1 - \gamma_A)(1 - \gamma_B)(q_A - q_B) + (1 - \gamma_B)\frac{\omega_A}{\beta_A} - (1 - \gamma_A)\frac{\omega_B}{\beta_B} > 0 \tag{8}$$

(notice the terms in all prices after $i+1$ cancel in the TIP comparison, and we divide through by λ_i).

Now divide through by $(1 - \gamma_A)(1 - \gamma_B)$ and this procedure yields our summary statistics:

A should be before B (in any consecutive pair, and hence in the global maximum) as long as

$$\Omega_A^\pi \equiv q_A + \frac{1}{(1 - \gamma_A)} \frac{\omega_A}{\beta_A} > q_B + \frac{1}{(1 - \gamma_B)} \frac{\omega_B}{\beta_B} \equiv \Omega_B^\pi$$

and so the TIP summary statistic is

$$\Omega_k^\pi \equiv q_k + \frac{1}{(1 - \gamma_k)} \frac{\omega_k}{\beta_k}$$

and firms should be ordered in decreasing order of these. Notice that this means (*ceteris paribus*):

- higher q 's earlier (these guys get more consumers with their high prices)
- higher $\frac{\omega_k}{\beta_k}$'s earlier (clear the decks of those who bring down the price a lot for all if they were late)
- higher γ earlier (less chance of success!): for profit you want the low (early) prices not to get much sales!!

3.2 Social welfare

now consider the pairwise ranking condition for Welfare – given firm pricing.

first note that any pair transposing does not affect welfare gained on EITHER earlier or later firms, given that consumers stop when they draw at least the medium valuation. thus we can look at a pair in isolation. And prices are just a transfer, and so do not enter the calculus.

So we first consider the surplus on searching A then B (conditional on having reached A at some position i), and compare with the converse.

The surplus is $q_A(1 - \gamma_A) + \beta_A \Delta_A$ earned on A plus the chance of not liking A and getting an analogous surplus on B , which also entails a search cost s . Adding this together

and using the analogous expression (switching subscripts) for the opposite order yields the condition for the sequence AB (for any pair) to be more profitable in aggregate than BA as:

$$\begin{aligned} & q_A(1 - \gamma_A) + \beta_A \Delta_A + \gamma_A(-s + q_B(1 - \gamma_B) + \beta_B \Delta_B) \\ > & q_B(1 - \gamma_B) + \beta_B \Delta_B + \gamma_B(-s + q_A(1 - \gamma_A) + \beta_A \Delta_A), \end{aligned}$$

which rearranges to

$$q_A(1 - \gamma_A) + \omega_A + \gamma_A(q_B(1 - \gamma_B) + \omega_B) > q_B(1 - \gamma_B) + \omega_B + \gamma_B(q_A(1 - \gamma_A) + \omega_A),$$

or

$$q_A(1 - \gamma_A)(1 - \gamma_B) + \omega_A(1 - \gamma_B) > q_B(1 - \gamma_B)(1 - \gamma_A) + \omega_B(1 - \gamma_A),$$

and hence

$$\Omega_A^W \equiv q_A + \frac{\omega_A}{(1 - \gamma_A)} > q_B + \frac{\omega_B}{(1 - \gamma_B)} \equiv \Omega_B^W$$

This gives the unique welfare-maximizing ranking condition as the decreasing order of the $\Omega_k^W \equiv q_k + \frac{\omega_k}{(1 - \gamma_k)}$.

For interpretation, big q 's are ranked early, *ceteris paribus*, because they deliver higher surplus earlier, and likewise for the surpluses on the high matches (the ω). Also, high γ are preferred earlier to get more shots at the High surplus.

We can compare to the order under TIP-maximization, where the summary statistic is $\Omega_k^\pi \equiv q_k + \frac{1}{(1 - \gamma_k)} \frac{\omega_k}{\beta_k}$. This puts weight on ω because of its effect on prices.

3.3 Consumer surplus

The consumer surplus case proceeds analogously to the welfare one, except now prices feature explicitly, and the q 's do not enter because they are priced out. The varying part (the later and earlier surpluses are unaffected by the order switch) of consumer surplus for the AB pair sequence with A in slot i and B in slot $i + 1$ is

$$(1 - \gamma_A) (q_A - p_A^i) + \beta_A \Delta_A + \gamma_A (-s + (1 - \gamma_B) (q_B - p_B^{i+1}) + \beta_B \Delta_B)$$

and the pricing rule gives $p_A^i = q_A - \frac{\omega_B}{\beta_B} - \kappa_{i+1}$ and $p_B^{i+1} = q_B - \kappa_{i+1}$ where $\kappa_{i+1} = \sum_{j>i+1} \frac{\omega_j}{\beta_j}$ denotes the sum of later price steps. Hence the consumer surplus difference of AB exceeds that of BA (which is found by transposing subscripts again) if

$$(1 - \gamma_A) \frac{\omega_B}{\beta_B} + \beta_A \Delta_A + \gamma_A (-s + \beta_B \Delta_B) > (1 - \gamma_B) \frac{\omega_A}{\beta_A} + \beta_B \Delta_B + \gamma_B (-s + \beta_A \Delta_A)$$

where the κ_{i+1} terms all cancel out: hence the same calculus applies regardless of which slot i is the base one.

Rearranging yields

$$\Omega_B^{CS} \equiv \frac{1}{(1 - \gamma_B)} \left(\frac{\omega_B}{\beta_B} - \omega_B \right) > \frac{1}{(1 - \gamma_A)} \left(\frac{\omega_A}{\beta_A} - \omega_A \right) \equiv \Omega_A^{CS}$$

so that the optimal order for the CS-maximizing criterion follows a *decreasing* order of the $\Omega_k^{CS} \equiv \frac{1}{(1 - \gamma_k)} \left(\frac{\omega_k}{\beta_k} - \omega_k \right) > 0$.

So A before B as $\gamma_B > \gamma_A$ which means more acceptable choices earlier, *ceteris paribus*.

The other term can be decomposed into two components, corresponding to price and surplus effects. First, a higher $\frac{\omega}{\beta}$ entails a higher price step and so should be placed later to keep consumers happier. Second, a higher ω means a higher surplus from the best match, *ceteris paribus*, and so should be placed earlier.

4 Slot/position auctions

Following previous literature we consider an allocation of the slots on an internet platform through an auction that assigns positions according to the ranking of bids (where higher bidders get earlier positions) and where a firm who wins a position is charged the next highest bid. Bids may be per click, per conversion (when the click is followed by a sale), or per position meaning that a firm pays for a position some lump sum amount.

We extend the search and competition model to have a number of firms, I , to exceed the number of positions $n \geq 1$. Hence, only the firms with the n highest bids get a slot and the

firm that is searched last, pays the $(n + 1)$ th bid. All other firms get some positive outside profit (which may differ across firms depending on the specification). The corresponding complete information auction game typically has multiple equilibria. We follow previous literature and impose an envy free condition (Edelman, Ostrovsky, and Schwartz, 2007) also called symmetry in Varian (2006) to refine the equilibrium concept. In those papers, the price paid by firms is per click. We present our framework in a way that allows different bases for charging the firms.

4.1 Symmetric firms

We start with an analysis of the auction in the symmetric firm case. Let $q_i = q$, $\Delta_i = \Delta$, $\gamma_i = \gamma$, $\alpha_i = \alpha$ and $\beta_i = \beta$ for all $i = 1, \dots, n$. The relevant pricing expression thus becomes

$$p_i = q - V_n - (n - i) \frac{\omega}{\beta}, \quad (9)$$

with $\omega = \beta\Delta - s$. Further assume that the outside profit earned by a firm that is not in any of the slots is identical for all I firms and denoted π_0 . We also assume that the value of V_n is unaffected by the outcome of the auction (which makes sense in the symmetric case).

First suppose that firms are charged per conversion. Previous papers (EOS and Varian, 2007) used "no-envy" to refine equilibria in asymmetric cases (a case discussed in the next subsection); Here we apply it under symmetry. It is a sufficient condition for equilibrium. It imposes the same condition for moving down in the order (higher i) and it considers deviations to earlier slots (lower i) that are more attractive than and cannot be achieved with a unilateral deviation: no envy imposes that firm i (In slot i) does not wish to deviate to a slot $j < i$, even if it could do so while being charged the same fee as what the firm in slot j is paying in equilibrium.

In the symmetric case it implies that all firms earn the same profit, and hence they all earn π_0 . Thus we can readily tie down equilibrium bids from this condition.

The profit of a firm in a sponsored link searched in position $i \leq n$ is, under per conversion

$$\pi_i^{conv} = \gamma^{i-1} (1 - \gamma) (p_i - b_{i+1}^{conv})$$

where b_{i+1}^{conv} is the $(i + 1)^{th}$ highest bid tendered. Setting $\pi_i^{conv} = \pi_0$ implies

$$b_{i+1}^{conv} = \left(p_i - \frac{\pi_0}{\gamma^{i-1} (1 - \gamma)} \right), i = 1, \dots, n$$

and b_1^{conv} simply has to exceed b_2^{conv} ; likewise b_{i+1}^{conv} must be less than b_n^{conv} for $i > n$.

We now want to find conditions under which the bid sequence is decreasing in i . For $i = 2, \dots, n$, we want

$$b_{i+1}^{conv} - b_i^{conv} = p_i - p_{i-1} - \frac{\pi_0}{\gamma} = \frac{\omega}{\beta} - \frac{\pi_0}{\gamma^{i-1}} \leq 0, \quad (10)$$

which is most stringent for $i = 2$ so a sufficient condition for equilibrium is $\frac{\omega}{\beta} \leq \frac{\pi_0}{\gamma}$. This condition requires that the outside profit π_0 is not too small. Indeed in order for firms to earn very low profits, bids per conversion (and hence per sale) must be very close to prices. However, since prices are increasing for slots further down the queue, bids cannot be decreasing as they should be in order to have an equilibrium. On the other hand, the condition is less stringent if γ is small, because then the benefit from being first in the search order is very large. Then firms in early slots may be making as much profit as firms in later slots, even though they are charging lower prices and paying higher fees to the platform. The condition that π_0 is large may be related to the level of the base quality q , to the extent that a large q guarantees large profits to all firms, whether they manage to obtain a sponsored link or not.

The analysis when firm are charged per click is identical to the per conversion case. Bids are lower but are paid with a higher probability so the platform's profit is identical. It requires however that firms are willing to bear more risk, because they pay for sure for each click but the prospect of converting the click into a sale remains uncertain.

If firms are charged per impression (just for being on the page and potentially seen by all who visit), profit may be written as

$$\pi_i^{imp} = (1 - \gamma) \gamma^{i-1} p_i - b_{i+1}^{imp}$$

and so

$$b_{i+1}^{imp} = (1 - \gamma) \gamma^{i-1} p_i - \pi_0$$

which is the difference between a firm’s gross profit in slot i and the outside profit. Bids are therefore decreasing in i if and only if gross profits are. Now gross profits are merely the product of the price and the probability of a sale. In order to have an equilibrium, we need that the percentage increase in price is less than the percentage decrease in the probability of a sale. Given the price expression (4), this is the case for q large enough. Note again that the condition on q large enough is less stringent for γ small so that the benefit from being searched early is large. Contrary to the per click and per conversion cases, here an equilibrium may exist even if $\pi_0 = 0$. Still, the platform’s expected profit is identical. However, this type of pricing requires that firms bear even more risk than with per click pricing.

4.2 Asymmetric firms

In the following analysis we allow for heterogeneity among firms in terms of base qualities q_i and incremental surpluses Δ_i keeping probabilities γ , α and β identical for different products. We still assume however that V_n is unaffected by the outcome of the auction.⁴ Hence the relevant pricing expression is, from (4),

$$p_i = q_i - V_n - \sum_{j=i+1}^n \frac{\omega_j}{\beta}. \quad (11)$$

It is convenient to introduce the additional notation p_i^j to denote the price charged by the firm who is in slot i in equilibrium if it moves to slot j while the order of all the other firms is preserved (although the position change for firm i results in a position change by one up or one down for firms that were in positions strictly between i and j , and also for firm j). Let B_i denote the equilibrium payment to the platform by the firm in position i (that is calculated from the $i + 1$ th highest bid). Finally, let λ_i be the number of clicks at slot i , which does not depend on the firms’ ranking because probabilities are the same across

⁴We do this mostly because it is not clear *a priori* what should be assumed about the exact way in which the identity of firms that do not win a slot affects search outside the sponsored links. Still, we expect that the insights from this section would not necessarily be affected much if we were more specific about search outside the sponsored links.

products. This is consistent with the setting in Varian (2006) and Edelman *et al.* (2007) but not with Chen and He (2011) or Athey and Ellison (2011).

The envy-free condition may be written as follows. For all $i, j = 1, \dots, n$

$$\lambda_i(1 - \gamma)p_i - B_i \geq \lambda_j(1 - \gamma)p_i^j - B_j \quad (12)$$

For $j > i$, this is equivalent to the Nash equilibrium condition that firm i should not wish to move to some slot j which is beyond its candidate equilibrium slot. For $j < i$, it implies that firm i does not want to move to some position j that precedes its own equilibrium position, but it is actually stronger. Condition (12) compares firm i 's equilibrium payoff to what it would get by switching to position j while paying the same amount to the platform as what firm j is paying in equilibrium. However, if $j < i$, firm i cannot achieve this payoff, because in order to capture firm j 's position, it needs to bid more than what firm j is bidding, which is $B_{j-1} \geq B_j$ and ends up paying B_{j-1} (where we denote B_0 the highest equilibrium bid to accommodate the case where $j = 1$).

To see the implications of the envy-free condition for the ranking of firms in our setting, we proceed as Varian (2006) and apply it both ways to two consecutive equilibrium positions. The condition that i does not envy $i + 1$ reads

$$\lambda_i(1 - \gamma)p_i - B_i \geq \lambda_{i+1}(1 - \gamma)p_i^{i+1} - B_{i+1} \quad (13)$$

and the condition that $i + 1$ does not envy i reads

$$\lambda_{i+1}(1 - \gamma)p_{i+1} - B_{i+1} \geq \lambda_i(1 - \gamma)p_{i+1}^i - B_i. \quad (14)$$

Subtracting (14) from (13) yields

$$\lambda_i(p_i - p_{i+1}^i) \geq \lambda_{i+1}(p_i^{i+1} - p_{i+1}). \quad (15)$$

Using the price formula (11) we may rewrite the above inequality as

$$\lambda_i(q_i - q_{i+1} + \Delta_i - \Delta_{i+1}) \geq \lambda_{i+1}(q_i - q_{i+1}), \quad (16)$$

Or equivalently,

$$(1 - \gamma)q_i + \Delta_i \geq (1 - \gamma)q_{i+1} + \Delta_{i+1}, \quad (17)$$

where we have used $\lambda_{i+1} = \gamma\lambda_i$.

Note that bids do not enter in the inequality (17). This is why the formula applies, independent of whether firm bid per click, per conversion or per impression. This also means that it puts no restriction on the equilibrium bids. We now discuss its implications for the ordering of firms, assuming that an envy-free equilibrium indeed exists, which requires that there exists bids that satisfy the envy-free restriction that implement the order. If base qualities are identical, the arguments from the symmetric case could be adapted to derive conditions for the existence of an equilibrium. If base qualities differ, but firms are ordered according to decreasing base qualities, then the conditions for existence are even less stringent.

First, condition (17) may readily be compared to the condition that ensures that social surplus is maximized, which is $(1 - \gamma)q_i + \beta\Delta_i \geq (1 - \gamma)q_{i+1} + \beta\Delta_{i+1}$. Hence the equilibrium order puts too much weight on the difference in niche values.

For $q_i = q_{i+1}$, (17) implies that $\Delta_i \geq \Delta_{i+1}$. This shows that *if all firms have the same base quality and probabilities are the same, then the no-envy outcome of the position auction ranks increments Δ_i in decreasing order. This coincides with the second-best socially optimal solution as well as the joint profit maximizing solution. It is however not the preferred solution for consumers.* (welfare properties are derived in the next section).

If $\Delta_i = \Delta_{i+1}$, then (17) implies $q_i \geq q_{i+1}$. It follows that *if firms have identical surplus increment over base quality and identical probabilities, then the no-envy outcome of the position auction ranks firms according to decreasing base qualities. This also maximizes social surplus as well as joint profit. Consumers are indifferent as to the order of firms.*

The model with identical values of Δ_i and identical probabilities coincides with Varian (2006) or Edelman *et al.* (2007) as far as the auction goes: the number of clicks at a given slots is independent of which firm takes which position and the value of a click to a firm in a given slot is a private value. Still, there are some important differences. Our model accounts

for the consumers' optimal search behavior as well as the firms' pricing. As a consequence, the value to a firm of a click in a given slot, although it is private, does depend on the number of subsequent slots whereas it is independent of the firm's position in those previous papers.

If we allow the values of Δ_i to differ across firms, then the value of a click in a given slot for firm i is no more private value because it depends on the type of the firms that are ranked after firm i . If base qualities are the same for all firms, then the envy free requirement still selects a unique ranking of firms from the largest to the lowest value of Δ_i .

Athey and Ellison (2011) as well as Chen and He (2011) highlight another source of externality that might alter the private value nature of the auction. In their setting, the probability that a consumer is at all interested in a product (which translates in our model into a strictly positive willingness to pay for the product) differs across firms. In order to obtain the Athey and Ellison auction environment in our setting, we would need to assume different values of γ_i while assuming β_i , Δ_i , and q_i are the same for all products. Still the search behavior of consumers would be different and as above, the value of a click for a firm in slots i would depend on how many slots are left after slot i . This value would however not depend on the characteristics of the firms positioned after i . They would only depend on the characteristics of firms that precede firm i as in Athey and Ellison (2011). But they also have that γ_i is private information to firm i , which modifies both the search behavior of consumers and the analysis of the auction, which is a Bayesian game.

The no-envy requirement may be rewritten to accommodate different match value probabilities, α_i , β_i and γ_i across products. It is no more the case however that the total amount paid in a given slot does not depend on the order when bidding is per click or per conversion. Even when bids are per impression, the no-envy conditions (12) must be modified to

$$\lambda_i(1 - \gamma_i)p_i - b_{i+1} \geq \lambda_i^j(1 - \gamma_i)p_i^j - b_{j+1}, \quad (18)$$

where λ_i^j is the number of clicks at slot j if firm i has replaced firm j in that slot. Applying no envy between two consecutive slots i and $i + 1$ yields

$$\lambda_i(1 - \gamma_i)p_i - b_{i+1} \geq \lambda_i\gamma_{i+1}(1 - \gamma_i)p_i^{i+1} - b_{i+2}, \quad (19)$$

and

$$\lambda_i \gamma_i (1 - \gamma_{i+1}) p_{i+1} - B_{i+1} \geq \lambda_i (1 - \gamma_{i+1}) p_{i+1}^i - B_i, \quad (20)$$

Prices are given by (4) so that, subtracting the second inequality from the first, we have that firms are now ordered according to the following condition:

$$(1 - \gamma_{i+1}) \left[(1 - \gamma_i) q_i + \frac{\omega_i}{\beta_i} \right] \geq (1 - \gamma_i) \left[(1 - \gamma_{i+1}) q_{i+1} + \frac{\omega_{i+1}}{\beta_{i+1}} \right] \quad (21)$$

This condition is equivalent to saying that the incremental value of being in slot i rather than in slot $i + 1$ is larger for Firm i than for Firm $i + 1$. This can be seen by rewriting it as

$$(1 - \gamma_i) \left[(1 - \gamma_{i+1}) q_i - \frac{\omega_{i+1}}{\beta_{i+1}} \right] \geq (1 - \gamma_{i+1}) \left[(1 - \gamma_i) q_{i+1} - \frac{\omega_i}{\beta_i} \right], \quad (22)$$

which can be obtained after some cancelations from comparing incremental profits. From our analysis of optimal rankings, this condition is also necessary and sufficient for the maximization of producer surplus for those firms who are in one of the n slots.

Next consider social surplus. As for producer surplus, we may isolate two consecutive slots. Maximization of social surplus for slots i and $i + 1$ requires

$$(1 - \gamma_{i+1})(1 - \gamma_i)(q_i - q_{i+1}) + [(1 - \gamma_{i+1})\omega_i - (1 - \gamma_i)\omega_{i+1}] \geq 0. \quad (23)$$

The joint profit maximizing solution (22) yields an analogous expression except that ω_i and ω_{i+1} are replaced by $\frac{\omega_i}{\beta_i}$ and $\frac{\omega_{i+1}}{\beta_{i+1}}$ respectively. This means that the potential distortions from the social optimum that arise with per impression bidding are caused by niche values rather than differences in base values. If probabilities γ_i differ, the only instance where the equilibrium order is socially optimal is if $\omega_i = 0$ for all i (to be contrasted with the case with identical match value probabilities where the social optimum was achieved by having either identical base qualities or identical niche values).

Additional insights can be obtained by considering the case where firms only differ in terms of the probability that the consumer has no interest in the product γ_i (where we presume they differ in the value of α_i so that they share the same niche probability $\beta_i = \beta$ for all i). Then, because $\omega_i = \omega_{i+1} = \omega > 0$, social surplus maximization merely requires

that $\gamma_i \geq \gamma_{i+1}$. This says that firms with the smallest potential market (smallest $(1 - \gamma_i)$) should be searched before the other firms. As a result, consumers will search more than if firms were ranked in the reverse order. This is socially desirable, because the firms' pricing induces insufficient search: consumers stop as soon as they have a strictly positive valuation for a product, whereas, $\omega > 0$ implies that they should search until they hold a valuation of $q + \Delta$. Hence the inefficiency stems from consumers holding q_i with the first firm choosing not to search. Having the product with the largest γ_i first makes this inefficiency smaller. This socially optimal ordering is however a second best result. In the first best solution that would prevail if all firms were charging the same price below q_i , because products share the same probability of a high match β_i and the same niche value $\Delta - i$, the order in which consumers search does not matter.

Now if we consider the equilibrium order of products resulting from a per impression auction, we find that as long as $\omega > 0$, it also satisfies $\gamma_i \geq \gamma_{i+1}$. The intuition is as follows. When moving from slot $i + 1$ to slot i , both firms increase their demand by the same amount $(1 - \gamma_i)(1 - \gamma_{i+1})$ and they both need to drop their price by $\frac{\omega}{\beta}$. However the inframarginal cost of this price decrease is proportional to the firm's consumer base in slot $i + 1$. Hence this cost will be smaller for a firm that has a smaller consumer base and hence a larger γ_i . As a result, its incremental profit is larger and it is ranked first in the no envy equilibrium. This order also maximizes total profit. Indeed, price being increasing in the search order when base qualities are identical, it is optimal that firms with a larger consumer base are searched last. Note however that the equilibrium outcome will typically leave out the firms with the largest consumer base (for reasonable specifications of π_o as a function of γ_i).

Alternatively, we can consider the case where firms have identical consumer bases and base qualities but different values for ω_i . Then the social optimum requires that products with the largest values for ω_i are searched first, The equilibrium ordering by contrast ranks firms with the largest value of $\frac{\omega_i}{\beta_i}$ first. We have already encountered the special case where β_i is identical for all firms, in which case the equilibrium order is socially optimal. This social efficiency of the equilibrium order also holds if products share the same niche value

$\Delta_i = \Delta$, because then, a larger ω_i is equivalent to a larger $\frac{\omega_i}{\beta_i}$. In order to get an inefficient ranking in equilibrium large niche values must be associated with low niche probabilities.

What about consumer surplus?

Let us now turn to per click bidding, which has been the main focus of previous literature. The no envy requirement then reads as follows:

$$\lambda_i ((1 - \gamma_i)p_i - b_i) \geq \lambda_i^j ((1 - \gamma_i)p_i^j - b_j), \quad (24)$$

where b_i denotes the equilibrium per click bid of the firm in slot i . Applying this condition both ways between slot i and $i + 1$ we must have

$$\lambda_i ((1 - \gamma_i)p_i - b_{i+1}) \geq \lambda_i \gamma_{i+1} ((1 - \gamma_i)p_i^{i+1} - b_{i+2}), \quad (25)$$

and

$$\lambda_i \gamma_i ((1 - \gamma_{i+1})p_{i+1} - b_{i+2}) \geq \lambda_i ((1 - \gamma_{i+1})p_{i+1}^i - b_{i+1}), \quad (26)$$

Again subtracting the second inequality from the first and using the pricing expression (4) we obtain the following ranking condition

$$(1 - \gamma_i) \left((1 - \gamma_{i+1})q_i - \frac{\omega_{i+1}}{\beta_{i+1}} \right) - \gamma_i b_{i+2} \geq (1 - \gamma_{i+1}) \left((1 - \gamma_i)q_{i+1} - \frac{\omega_i}{\beta_i} \right) - \gamma_{i+1} b_{i+2} \quad (27)$$

Note that if $b_{i+2} = 0$ or $\gamma_i = \gamma_{i+1}$, then this condition is the same as that for per impression bidding. Bid b_{i+2} should not be zero in a no envy equilibrium (in order to get increasing bids, we probably need that a firm's gross profit is larger if it is in an earlier slot, so it would take an earlier slot if it could get it for free - not completely clear this is true for instance with different q_i s: then the last firm in might be paying zero: but for the last slots, it depends a lot on the π_o specification). The case with $\gamma_i = \gamma_{i+1}$ is a generalization of our result with identical probabilities that the order is independent of the bid format. If $\gamma_i \neq \gamma_{i+1}$ then The order does not necessarily coincide with the joint profit maximizing order.

An illustration of this is the case where products only differ in the value of γ_i . Then the equilibrium order is characterized by the following simple condition:

$$(\gamma_i - \gamma_{i+1}) \left(\frac{\omega}{\beta} - b_{i+2} \right) \geq 0. \quad (28)$$

Then if the price difference is small enough, then the equilibrium order is such that $\gamma_i < \gamma_{i+1}$ so that products with a large consumer base are searched first. This is the case picked up by Athey and Ellison (2011) where $\frac{\omega}{\beta} = 0$. However in our setting, in contrast with their finding, this order is neither joint profit maximizing nor socially optimal. In their setting with heterogenous search costs and identical prices, this order encourages consumers to search longer rather than dropping out, which is good both for profits and social welfare. Here, qualities being equal, prices are larger at later slots so large sellers should be in those slots. Besides, having large sellers first would mean that more people stop search holding only q whereas it is socially optimal for them to search on until they obtain the highest possible match $q + \Delta$.

Similarly we may derive the equilibrium order with no envy for per conversion bids. The no envy requirement then reads as follows:

$$\lambda_i(1 - \gamma_i)(p_i - b_i) \geq \lambda_i^j(1 - \gamma_i)(p_i^j - b_j), \quad (29)$$

where b_i denotes the equilibrium per conversion bid of the firm in slot i . Applying this condition both ways between slot i and $i + 1$ we must have

$$\lambda_i(1 - \gamma_i)(p_i - b_i) \geq \lambda_i\gamma_{i+1}(1 - \gamma_i)(p_i^{i+1} - b_{i+1}), \quad (30)$$

and

$$\lambda_i\gamma_i(1 - \gamma_{i+1})(p_{i+1} - b_{i+1}) \geq \lambda_i(1 - \gamma_{i+1})(p_{i+1}^i - b_i), \quad (31)$$

Again subtracting the second inequality from the first and using the pricing expression (4) we obtain the following ranking condition

$$(1 - \gamma_{i+1})q_i - \frac{\omega_{i+1}}{\beta_{i+1}} - \gamma_i \left(\sum_{j=i+2}^n \frac{\omega_j}{\beta_j} + b_{i+2} \right) \geq (1 - \gamma_i)q_{i+1} - \frac{\omega_i}{\beta_i} - \gamma_{i+1} \left(\sum_{j=i+2}^n \frac{\omega_j}{\beta_j} + b_{i+2} \right) \quad (32)$$

(Preliminary, incomplete and inaccurate...)

References

- [1] Agarwal, A., Hosanagar, K., and Smith, M. (2008): Location, location, location: An analysis of profitability of position in online advertising markets. *Journal of Marketing Research*.
- [2] Anderson, S. P., A. de Palma and J. F. Thisse (1992): *Discrete Choice Theory of Product Differentiation*, MIT Press: Cambridge, MA, USA.
- [3] Arbatskaya, Maria (2007): Ordered search. *RAND Journal of Economics*, 38(1), 119-126.
- [4] Armstrong, Mark and Jidong Zhou (2011): Paying for Prominence. *Economic Journal*, 121, F368-F395.
- [5] Armstrong, M., and Zhou, J. (2010): Conditioning prices on search behaviour.
- [6] Armstrong, Mark, John Vickers, and Jidong Zhou (2009): Prominence and consumer search. *RAND Journal of Economics*, 40(2), 209-233.
- [7] Athey, Susan, and Glenn Ellison (2011): Position Auctions with Consumer Search. *Quarterly Journal of Economics*, 126(3), 1213-1270.
- [8] Baye, Michael R. and John Morgan (2001): Brand and Price Advertising in Online Markets. *Management Science*, 55, 1139-1151.
- [9] Baye, Michael R. and John Morgan (2009): Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets. *American Economic Review*, 91, 454-474.
- [10] Camera, G., and Selcuk, C. (2009): Price dispersion with directed search. *Journal of the European Economic Association*, 7(6), 1193-1224.
- [11] Chen, Yongmin and Chuan He (2011): Paid Placement: Advertising and Search on the Internet. *Economic Journal*, 121, F309-F328.

- [12] De Cornière, Alexandre (2013): Search Advertising. Mimeo, Oxford University.
- [13] Edelman, Ben and Michael Schwarz (2010): Optimal Auction Design and Equilibrium Selection in Sponsored Search Auctions. *American Economic Review*, 100.
- [14] Edelman, Ben, Michael Ostrovsky, and Michael Schwarz (2007): Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. *American Economic Review*, 97(1), 242-259.
- [15] Haan, Marco A. and José Luis Moraga-González (2011): Advertising for Attention in a Consumer Search Model. *Economic Journal*, 121, 552-579.
- [16] Haan, Marco A., José Luis Moraga-González and Vaiva Petrikaite (2013): Advertising, Consumer Search and Product Differentiation. Mimeo, University of Groningen.
- [17] Janssen, Maarten C.W. and José Luis Moraga-Gonzalez, and Wildenbeest, Matthijs R. (2005): Truly costly sequential search and oligopolistic pricing. *International Journal of Industrial Organization*, 23(5-6), 451-466.
- [18] Janssen, Maarten C. W. and José Luis Moraga-González (2004): Strategic Pricing, Consumer Search and the Number of Firms. *Review of Economic Studies*, 71(4), 1089-1118.
- [19] Moraga-Gonzalez, Jose Luis (2000): Quality Uncertainty and Informative Advertising. *International Journal of Industrial Organization*, 18, 615-640.
- [20] Moraga-Gonzalez, Jose Luis and Vaiva Petrikaite (2012): Search Costs, Demand-Side Economies and the Incentives to merge under Bertrand Competition. Tinbergen Institute Discussion Papers 12-017/1.
- [21] Robert, Jacques and Dale Stahl (1993): Informative Price Advertising in a Sequential Search Model. *Econometrica*, 61, 657-686.

- [22] Stahl, Dale (1994): Oligopolistic Pricing and Advertising. *Journal of Economic Theory*, 64, 162-177.
- [23] Varian, Hal R. (2007): Position auctions. *International Journal of Industrial Organization*, 25(6), 1163-1178.
- [24] Weitzman, Martin L. (1979): Optimal Search for the Best Alternative. *Econometrica*, 47(3), 641-54.
- [25] Wilson, Chris M. (2010): Ordered search and equilibrium obfuscation. *International Journal of Industrial Organization*, 28(5), 496-506.
- [26] Xu, L., Chen, J., and Whinston, A. (2010): Oligopolistic pricing with online search. *Journal of Management information systems*, 27(3), 111-142.
- [27] Zhou, Jidong (2011): Ordered search in differentiated markets. *International Journal of Industrial Organization*, 29(2), 253-262.