Low-Frequency Fiscal Uncertainty*

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Abstract

Fiscal variables' steady-state values, or synonymously, fiscal targets, are usually calibrated to historical means and assumed to be known to households inside the economy. This paper investigates effects of low-frequency fiscal uncertainty in an incomplete information, anticipated utility environment in which households are learning unknown fiscal targets. Highly persistent fiscal movements cause households to suspect fiscal targets may be time-varying, even though the underlying targets are all assumed time-invariant. An RBC model with a detailed fiscal section that features government spending, lump-sum transfers, risk-free debt and distortionary taxes is fit to US data to estimate structural parameters. Small deviations of households' beliefs on fiscal targets generate vastly different dynamics on how shocks influencing the real economy. Ignoring low-frequency fiscal uncertainty could lead to qualitatively and quantitatively misleading policy evaluations across all horizons. Using a Rao-Blackwellized particle filter, data suggest households' perceived fiscal targets vary and deviate from calibrated historical means.

Keywords: Incomplete Information; Fiscal policy; Anticipated utility; Bayesian estimation; Particle filtering;

JEL Classification Numbers: C11; D83; E32; E62; H68;

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1 Introduction

A common exercise in fiscal policy studies is to set the steady-state values of fiscal variables to their historical means or some exogenous values. Examples are ample and cover a large variety of topics including optimal policy, monetary and fiscal interactions and fiscal multipliers. Households *inside* the economic models are assumed to know the steady-state values of fiscal variables perfectly and policy uncertainties are modelled around either contemporaneous/future shocks or financing decisions in the short or medium run. In reality *no one*[1] knows these fiscal variables' steady-state values, which not only govern the instruments' behaviour (in consequence, the steady state of the economy) in the long run, but also influence the private sectors(i.e., firms and households)' decisions at both business-cycle or even on a daily basis. Not knowing fiscal variables' steady-state values, or equivalently, the fiscal targets, introduces prevalent uncertainties that "operate at all frequencies"(Leeper (2015)). I call uncertainties originated from unknown fiscal targets *"low-frequency fiscal uncertainty"*.

While the econometricians and economists outside the models can always bypass their uncertainties to fiscal targets by calibration and imposing additional structures(e.g., assuming fiscal targets are all time-invariant or follow certain discrete-time Markov chains), as decision makers households inside the economy have to confront low-frequency fiscal uncertainty directly. In this paper I consider an incomplete information environment in which households need to form and update beliefs on unknown fiscal targets before making consumption and saving decisions each period. Without perfect information, not well anchored beliefs open up the possibility that households' perceived fiscal targets may be different than the actual ones underlying the economy.

At the same time, fiscal variables are highly persistent. In many estimated DSGE models in which fiscal targets are calibrated to historical means and fiscal variables are modelled as stationary AR(1) processes, the autoregressive parameters of fiscal variables are very close to unity(e.g., Fernndez-Villaverde et al. (2015), Leeper et al. (2015)). Such persistence can be conveniently seen

[1] That also includes policy makers who need to design and implement the fiscal policy, the economists who build the models and the econometricians who take the models to data.

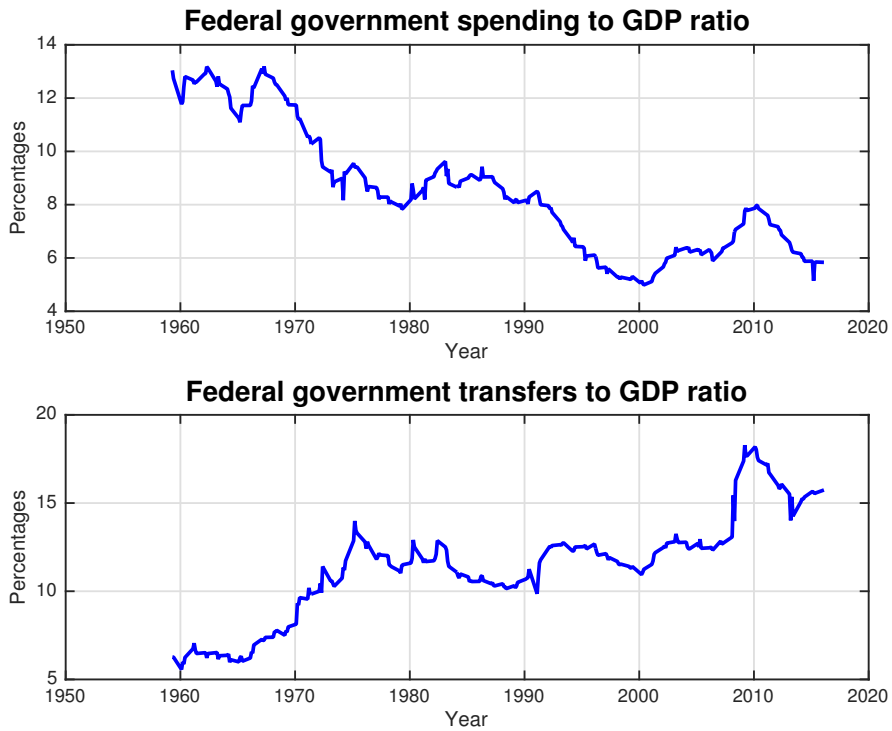

Figure 1: Historical US Federal government spending and transfers to GDP ratios. Data are from the Bureau of Economic Analysis's NIPA tables.

from realized fiscal series. Figure 1 plots the historical US federal government spending and transfers to GDP ratios from 1959Q3 to 2016Q1. These series appear to have their own "trends".[2] Slow-moving fiscal series raises the possibility that households may suspect fiscal variables' steady-state values are time-varying, given there are no explicitly announced fiscal targets and their beliefs are not anchored. Low-frequency fiscal movements exacerbate low-frequency fiscal uncertainty.

I test such possibility in an RBC model with a rich fiscal sector featuring government spending, lump-sum transfers, risk-free debt and distortionary(i.e., consumption, capital and labor) taxes. Following the common

[2]The word "trends" is in quotation marks since they cannot last forever. For instance, both series in Figure 1 should be bounded between [0,1].

exercise in practise, I model the actual fiscal targets as time-invariant and calibrate them to historical means. Households don't have perfect information on targets and need to learn them before forming expectations and making decisions. Due to highly persistent fiscal movements, a crucial misspecification households made is they suspect fiscal targets are time-varying. A signal extraction problem is established where households use past realizations of fiscal variables and Kalman filters to infer their perceived fiscal targets. Households then form expectations conditional on the perceived fiscal targets using anticipated utility(Kreps (1998)). I call the incomplete information, anticipated utility environment IIAU. I explore consequences of low-frequency fiscal uncertainty by comparing the equilibrium dynamics in the IIAU model with the full information, rational expectation(FIRE) model.

Recent literature highlights policy uncertainties by looking at stochastic volatilities of policy instruments(e.g. Born and Pfeifer (2014)). Fernndez-Villaverde et al. (2015) estimates tax and spending processes for the United States in which the volatility of fiscal policy is assumed to be changing over time. Precautionary saving motives raised by time-varying volatilities decrease consumption and an endogenous increase in markups depresses investment, leading to detrimental effects on the real economy. These effects are reinforced when monetary policy is stuck at the zero lower bound. I follow Hollmayr and Matthes (2015) and Richter and Throckmorton (2015) and consider policy uncertainties which directly influence households' beliefs of the economy's steady states. In contrast to Hollmayr and Matthes (2015) who simulates costs of fiscal uncertainty by considering a one-time policy change and let households learn all parameters in the new fiscal rules, I focus on low-frequency fiscal uncertainty by designing an environment where households only need to learn unknown fiscal targets. In Hollmayr and Matthes (2015) the learning process is fast and households quickly learn the new fiscal rule and the model dynamics converge to FIRE after several quarters. In my model households are always learning due to their suspicions of time-varying fiscal targets. This design also allows me to take the model to data. Richter and Throckmorton (2015) calibrates the debt targets to an exogenous discrete-time Markov chain and let households learn both the state and transition matrix of the Markov chain. In my model all actual fiscal targets are time-invariant and it is the households' misperception that fiscal targets are time-varying generates ever-changing perceived steady states that constantly deviate from the underlying actual fiscal targets.

The main contribution of this paper is twofold: (i) First, I use a simple analytical model to illustrate the propagation mechanism of low-frequency fiscal uncertainty. The analytical calculations also make clear why not knowing fiscal targets introduces prevalent uncertainties and highlight the importance of households' beliefs on fiscal targets. In contrast to the FIRE case where *i.i.d.* tax shocks have no impact on the real economy, introducing low-frequency uncertainty and varying households' beliefs on tax targets can generate upward, flat, or downward impulse responses. (ii) Second, I assess the effects of low-frequency uncertainties by connecting an RBC model with data. I incorporate the effects of households' beliefs by establishing the IIAU model solution as a conditionally linear state transition equation with a set of expanded state variables. Deep structural parameters of the model, including those which govern the evolution of households' beliefs are estimated by using the Bayesian approach and a Rao-Blackwellized particle filter is applied to the IIAU model to extract households' historical beliefs on fiscal targets.

The main result of the paper is that small deviations of households' beliefs of fiscal targets could generate vastly different dynamics on how shocks influencing the real economy. For instance, in contrast to the FIRE impulse responses which indicate government spending is expansionary in the short run and is slightly contractionary in the long run, different beliefs about fiscal targets implies in the long run the same government shock can become either highly expansionary or highly contractionary. In the FIRE model a temporary increase in capital tax depresses investment and forces households to substitute more saving for consumption, causing a mild contraction in the output in the short run. Under slightly different beliefs in the IIAU model the same capital tax shock however can become highly expansionary or contractionary in the short run and such effects can last several decades. These results are illustrated by plotting impulse responses and tracking the evolution of households' beliefs: In resonance with results found in the analytical model, impulse responses of variables converge to their perceived steady states first, along which they gradually converge to their actual steady-state values provided there are no subsequent shocks. Households' beliefs on fiscal targets thus play crucial roles and simply ignoring low-frequency fiscal uncertainty may lead to qualitatively and quantitatively misleading policy evaluations across all horizons. To gather evidence of variations of house-

holds' perceived fiscal targets, I utilize the conditional linearity of the IIAU model and extract households' historical beliefs on U.S. fiscal targets using a Rao-Blackwellized particle filter(Schon et al. (2005)). Results suggest that there are large deviations(10% to 30%) of households' beliefs from calibrated values. Using the extracted beliefs for 2007Q3, impulse responses imply low-frequency fiscal uncertainty may deepen the Great Recession.

The paper proceeds as follows. In Section 2 I solve a simple growth model analytically to lay out the transmission mechanism. Section 3 describes an RBC model with a rich fiscal sector which will be taken to the data. Section 4 characterizes the households' information sets and defines the households' signal extraction problem on fiscal targets. Section 5 estimates the structural parameters of the RBC model both for the FIRE model and a linearized IIAU model. I then illustrate the effects of low-frequency fiscal uncertainty and emphasize the importance of households' beliefs by plotting impulse responses. A Rao-Blackwellized particle filter is then applied to the nonlinear IIAU model to extract households' historical perceived fiscal targets. Section 6 concludes.

2 Analytical Example

An analytical example helps to illustrate the underlying mechanism and consequences of low-frequency fiscal uncertainty. I consider a standard neoclassical growth model with CRRA preferences, complete depreciation of capital and a set of simple fiscal policies.

Consider an economy with a unit measure of identical households who choose a sequence of consumption and capital $\{C_t, K_t\}_{t=0}^{\infty}$ to maximize expected CRRA utility $E_0^{\mathcal{I}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}$, where $\beta \in (0,1)$ is the subjective discount factor and γ measures the degree of relative risk aversion. $E_t^{\mathcal{I}}$ is an expectation operator conditional on the representative household's information set $\mathcal{I}_t$(defined in section 2.1). A perfectly competitive firm produces output Y subject to a Cobb-Douglas technology $Y_t = A_t K_{t-1}^{\alpha}$ with $\alpha \in (0,1)$. Capital depreciates fully after one period. Households pay income taxes τY and receive lump-sum transfers T from the government. Household choices are constrained by $C_t + K_t \leq (1-\tau_t) A_t K_{t-1}^{\alpha} + T_t$. Technological progress

follows a stationary AR(1) process[3]

$$\log(A_t) = \rho \log(A_{t-1}) + \sigma_A \epsilon_t^A, \quad \epsilon_t^A \sim N(0,1) \tag{1}$$

with $\rho \in [0,1)$. Government runs a balanced budget each period, $T_t = \tau_t Y_t$, where the tax rate τ_t follows a simple *i.i.d.* rule

$$\log(\tau_t) = \log(\tau^*) + \sigma_\tau \epsilon_t^\tau, \quad \epsilon_t^\tau \sim N(0,1) \tag{2}$$

and τ^* is the steady-state income tax rate chosen by the fiscal authority.

I make the following timing assumptions: First, the economy enters time period t with K_{t-1} inherited from last period's households' savings decision. Shocks $\epsilon_t^A, \epsilon_t^\tau$ are realized, which determines the tax rate according to (2), output, and therefore tax revenue. Transfers adjust to satisfy the balanced-budget constraint. Households then make consumption/savings decisions and the period ends.

The equilibrium conditions of the model consist of the inter-temporal Euler equation (3) and aggregate resource constraint (4)

$$\frac{1}{C_t^\gamma} = \alpha\beta E_t^{\mathcal{I}}[(1-\tau_{t+1})\frac{A_{t+1}K_t^{\alpha-1}}{C_{t+1}^\gamma}], \tag{3}$$

$$C_t + K_t = Y_t = A_t K_{t-1}^\alpha. \tag{4}$$

I consider two information sets: a standard complete-information case that is consistent with rational expectations and incomplete information, where households must learn about the unknown steady-state tax rate τ^*.

2.1 Households' information sets $\mathcal{I}_t$

Households completely understand the model structure $\mathcal{M}$. In particular, besides acting as price takers and making consumption and saving decisions, households are aware of the balanced government budget constraint and the AR(1) technology (1) and *i.i.d* tax rule (2). Before choosing C_t and K_t, they observe the entire history of observables

$$\mathcal{O}_t = \{C^{t-1}, K^{t-1}, \tau^t, A^t, T^t, Y^t\}$$

[3]Steady-state technology level A^* thus has been normalized to 1.

where $X^t = \{X_t, X_{t-1}, ...\}$ denotes the history of variable X up to time t.[4] I denote parameters of the model as

$$\mathcal{P} = \{\beta, \gamma, \alpha, \rho, A^*, \tau^*, \sigma_A, \sigma_\tau\} = \mathcal{P}_1 \cup \{\tau^*\}$$

where I separate τ^* from $\mathcal{P}_1 = \{\beta, \gamma, \alpha, \rho, A^*, \sigma_A\}$ to allow the possibility households do not have direct knowledge of τ^*.

The history of exogenous shocks are denoted by $\mathcal{S}_t^A = \{\epsilon_0^A, \epsilon_1^A, ..., \epsilon_t^A\}$, $\mathcal{S}_t^\tau = \{\epsilon_0^\tau, \epsilon_1^\tau, ..., \epsilon_t^\tau\}$, and are not directly observable by households. However, once $\{A^t\}$(which belongs to $\mathcal{O}_t$), $\{\rho, \sigma_A\}$(which is a subset of $\mathcal{P}$), and the AR(1) technology (1)(which is part of the model structure $\mathcal{M}$) are observed by households, they can figure out $\mathcal{S}_t^A$ perfectly due to the invertibility of the stationary AR(1) process. Similarly, they can unravel $\mathcal{S}_t^\tau$ if $\{\tau^*, \sigma_\tau\}$, $\{\tau^t\}$ and the *i.i.d.* tax rule (2) are in $\mathcal{I}_t$.

In section 2.2, I consider two information sets:

$$\begin{aligned} \mathcal{I}_t^{Full\ Info.} &= \{\mathcal{M}; \mathcal{O}_t; \mathcal{P}\}; \\ \mathcal{I}_t^{Incomp.\ Info.} &= \{\mathcal{M}; \mathcal{O}_t; \mathcal{P}_1\} = \mathcal{I}_t^{Full\ Info.} \setminus \{\tau^*\}. \end{aligned}$$

The former $\mathcal{I}_t$ corresponds to a full information case where households have perfect knowledge of the model structure $\mathcal{M}$, parameters $\mathcal{P}$, observables $\mathcal{O}_t$, and history of shocks $\{\mathcal{S}_t^A, \mathcal{S}_t^\tau\}$. The latter $\mathcal{I}_t$ corresponds to an incomplete information scenario where the steady-state tax rate τ^* is unknown to households. Even though $\mathcal{S}_t^A$ is still in $\mathcal{I}_t^{Incomp.\ Info.}$, not knowing τ^* implies $\mathcal{S}_t^\tau$ is not necessarily in $\mathcal{I}_t^{Incomp.\ Info.}$.

Under full information, households form expectations using rational expectation(RE) and I denote the conditional expectation operator $E_t^{\mathcal{I}}$ as $\mathbb{E}_t$ and refer to this case as FIRE (full information, rational expectations). Solutions to this case are well known and will be treated as a benchmark.

Under incomplete information households first need to form a belief about τ^*. They then adhere to the anticipated utility approach pioneered by Kreps (1998) and introduced into a macro setting by Cogley and Sargent (2008). For

[4] The initial time index of X^t is variable-dependent and could start either from 0 or -1. For example, $C^t = \{C_t, C_{t-1}, ..., C_0\}$ while $K^t = \{K_t, K_{t-1}, ..., K_0, K_{-1}\}$.

abbreviation, I call the corresponding incomplete information, anticipated utility model IIAU. Let the expectation operator $E_t^{\mathcal{I}}$ be denoted $E_t^{\circledast}$ under incomplete information. During each period, households maximize expected utility conditional on their own beliefs of the steady-state tax rate, denoted by $\tau_t^{\circledast}$. The subscript of $\tau_t^{\circledast}$ indicates households' beliefs could change over time.

2.2 Model Solutions

It suffices to focus on the capital level K_t to solve the model. Substituting C_t in (3) using (4) yields

$$\frac{1}{(A_t K_{t-1}^{\alpha} - K_t)^{\gamma}} = \alpha\beta E_t^{\mathcal{I}} \left[(1-\tau_{t+1}) \frac{A_{t+1} K_t^{\alpha-1}}{(A_{t+1} K_t^{\alpha} - K_{t+1})^{\gamma}} \right]. \tag{5}$$

The FIRE steady state capital stock $K^* = [\alpha\beta(1-\tau^*)]^{1/(1-\alpha)}$ is time-invariant while the IIAU steady state capital stock $K_t^{\circledast} = [\alpha\beta(1-\tau_t^{\circledast})]^{1/(1-\alpha)}$ could display fluctuations over time due to changes in $\tau_t^{\circledast}$. Simple algebra shows $K_t^{\circledast}/K^* = [(1-\tau_t^{\circledast})/(1-\tau^*)]^{1/(1-\alpha)}$. Thus, the elasticity of $K_t^{\circledast}/K^*$ with respect to $(1-\tau_t^{\circledast})/(1-\tau^*)$ is $1/(1-\alpha) > 1$. Given a capital share $\alpha = 1/3$, if $(1-\tau_t^{\circledast})$ is 1% lower than $(1-\tau^*)$[5], then $K_t^{\circledast}$ will be 1.5% lower than K^*.

Let $\hat{X}_t = \log(X_t) - \log(X^*)$ denote percentage deviations from steady state values under FIRE and $\widetilde{X}_t = \log(X_t) - \log(X_t^{\circledast})$ be the counterpart under IIAU, log-linearizing (5) produces a second-order difference equation in capital,

$$\begin{aligned}
\text{FIRE}: \ & \alpha\beta\gamma(1-\tau^*)\mathbb{E}_t\hat{K}_{t+1} + \big[(\alpha-1)(1-\alpha\beta(1-\tau^*)) \\
& -\alpha\gamma - \alpha\beta\gamma(1-\tau^*)\big]\hat{K}_t + \alpha\gamma\hat{K}_{t-1} = \\
& \big[\gamma - 1 + \alpha\beta(1-\tau^*)\big]\mathbb{E}_t\hat{A}_{t+1} - \gamma\hat{A}_t + \frac{\tau^*}{1-\tau^*}\big[1-\alpha\beta(1-\tau^*)\big]\mathbb{E}_t\hat{\tau}_{t+1} \qquad (6) \\
\text{IIAU}: \ & \alpha\beta\gamma(1-\tau_t^{\circledast})E_t^{\circledast}\widetilde{K}_{t+1} + \big[(\alpha-1)(1-\alpha\beta(1-\tau_t^{\circledast})) \\
& -\alpha\gamma - \alpha\beta\gamma(1-\tau_t^{\circledast})\big]\widetilde{K}_t + \alpha\gamma\widetilde{K}_{t-1} = \\
& \big[\gamma - 1 + \alpha\beta(1-\tau_t^{\circledast})\big]E_t^{\circledast}\widetilde{A}_{t+1} - \gamma\widetilde{A}_t + \frac{\tau_t^{\circledast}}{1-\tau_t^{\circledast}}\big[1-\alpha\beta(1-\tau_t^{\circledast})\big]E_t^{\circledast}\widetilde{\tau}_{t+1} \qquad (7)
\end{aligned}$$

[5] A concrete example would be $\tau^* = 0.3$ while $\tau_t^{\circledast} = 1-(1-0.3)/1.01 \approx 0.307$.

Comparing (6) and (7) illustrates the differences between the two approaches: $\mathbb{E}_t$ compared to $E_t^{\circledast}$, and $\hat{X}_t$ to $\widetilde{X}_t$ for $X = \{K, A, \tau\}$ when moving from FIRE to IIAU, all τ^*'s in (6) have been replaced with $\tau_t^{\circledast}$ in (7).[6] This is due to the underlying assumption of how households form expectations in the anticipated utility approach: Households update their beliefs on tax targets using Bayes' law but optimize myopically. The anticipated utility approach thus encodes a form of bounded rationality but abstracts from the private agents' precautionary motive driven by uncertainty to future beliefs on steady-state tax rates. This property allows us to treat $\tau_t^{\circledast}$ as a constant when solving (7). The second-order linear difference equation thus can be solved by using well-known linear solution methods.

Since the $i.i.d$ tax rule (2) is in both $\mathcal{I}_t^{Full\ Info.}$ and $\mathcal{I}_t^{Incomp.\ Info.}$, households must know next period's tax deviation is unpredictable under both FIRE and IIAU. It follows that $\mathbb{E}_t\hat{\tau}_{t+1} = 0$ in (6) and $E_t^{\circledast}\widetilde{\tau}_{t+1} = 0$ in (7). Appendix A.1 shows solutions to (6) and (7) are of the form

$$\text{FIRE}: \hat{K}_t = f_1(\tau^*)\hat{K}_{t-1} + f_2(\tau^*)\hat{A}_t \tag{8}$$

$$\text{IIAU}: \widetilde{K}_t = f_1(\tau_t^{\circledast})\widetilde{K}_{t-1} + f_2(\tau_t^{\circledast})\widetilde{A}_t \tag{9}$$

where I express the response coefficients of $\hat{K}_t$ (or $\widetilde{K}_t$) to $\hat{K}_{t-1}$ (or $\widetilde{K}_{t-1}$) and $\hat{A}_t$ (or $\widetilde{A}_t$) as functions of τ^*(or $\tau_t^{\circledast}$). While in the FIRE model τ^* is simply a fixed parameter, in the IIAU model, $\tau_t^{\circledast}$ becomes an *additional* state variable and enters the policy function *non-linearly*.

It is important to emphasize different roles τ^* and $\tau_t^{\circledast}$ played in the IIAU model. Calculations of the steady state capital level and policy functions (8), (9) indicate households' beliefs $\tau_t^{\circledast}$ are used to update the steady state tax rate τ^*, and therefore enter the system as an additional state variable. The perceived fiscal target $\tau_t^{\circledast}$ not only influences the steady state of the economy, but also enters policy functions non-linearly.

These effects are wide-spread: although I only focus on the capital level K_t, it is easy to see all other endogenous variables (i.e., C_t, Y_t and T_t) will be influenced by households' beliefs about steady-state tax rate. The actual tax target τ^* now plays an implicit role in governing the evolution of $\tau_t^{\circledast}$. In particular, along with the $i.i.d$ tax rule (2) and shocks ϵ_t^{τ}, τ^* determines the true

[6] Since $\mathcal{S}_t^A$ belongs to both $\mathcal{I}_t^{Full\ Info.}$ and $\mathcal{I}_t^{Incomp.Info.}$, it follows $\hat{A}_t = \widetilde{A}_t$ for all t.

data generating process of tax rates, τ_t, which directly impacts how households form their beliefs about $\tau_t^{\circledast}$. This highlights multiple roles τ_t plays in the IIAU model: Besides acting as realizations of the fiscal instrument which finances lump-sum transfers, τ_t's also serve as "signals" and shape household's opinions on τ^*. With unknown τ^* and only observing the realized tax rates, even this *i.i.d.* tax rule (2) creates a meaningful signal extraction problem for the households.

The IIAU capital policy function (9) indicates how households update beliefs $\tau_t^{\circledast}$ is crucial in transmitting the impacts of not knowing the actual tax target τ^*. Since households know the linear tax rule (2), I follow Hollmayr and Matthes (2015) and assume they choose the optimal recursive filter for linear systems and use Kalman filter to update beliefs. The following state-space model(SSM) establishes the household's learning problem on τ^*

$$\text{State equation:} \qquad \log(\tau_t^{\circledast}) = \log(\tau_{t-1}^{\circledast}) + \sigma_\eta \eta_t, \quad \eta_t \sim N(0,1), \tag{10}$$

$$\text{Observation equation:} \qquad \log(\tau_t) = \log(\tau_t^{\circledast}) + \sigma_\tau \epsilon_t^\tau, \quad \epsilon_t^\tau \sim N(0,1). \tag{11}$$

The state equation (10) implies households believe the tax target τ^* is time-varying($\sigma_\eta > 0$). In the following empirical sections this misperception originates from the facts that households' beliefs on fiscal targets are not anchored and fiscal variables are highly persistent. The perceived time-varying fiscal targets assumption delivers perpetual learning as households always believe there might be some structural changes to the underlying tax targets. Only when $\sigma_\eta = 0$ and $\tau_t^{\circledast} \equiv \tau^*$ the above SSM nests the true data generating process (2).

The FIRE capital policy function (8) indicates the impulse response functions(IRFs) of $\hat{K}_t$ with respect to ϵ_t^τ-shocks are identically 0.[7] In the IIAU environment, since ϵ_t^τ shocks alter observed tax rats τ_t, they will influence how households form their beliefs $\tau_t^{\circledast}$, which will impact the real economy. In contrast to the FIRE, in the IIAU model capital level will have some *non-trivial* impulse response functions.

Figure 2 plots impulse responses to a one standard deviation shock in tax rates for various initial beliefs on tax target when the initial capital level is on its deterministic steady state K^*. The initial beliefs on tax targets, $\tau_1^{\circledast}$,

[7] This is, of course, due to the *i.i.d* tax rule so that $\mathbb{E}_t \hat{\tau}_{t+1} = 0$.

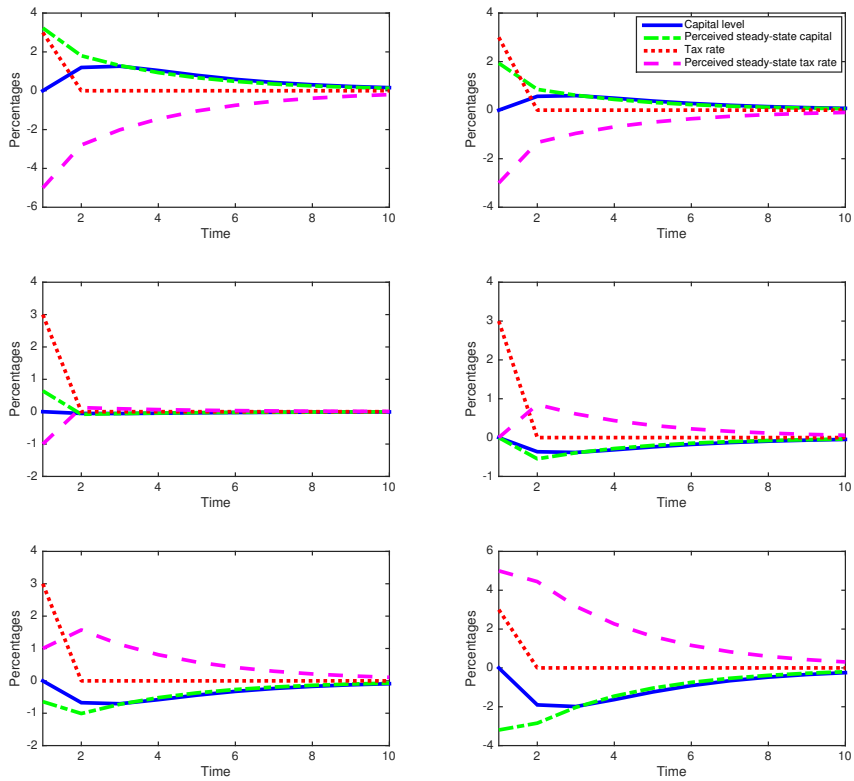

Figure 2: Impulse Responses to a one standard deviation in tax shock by varying initial beliefs on tax target. Dotted Red: Observed tax rates; Dashed Purple: Perceived tax targets; Solid Blue: Capital; Dash-dotted Green: Perceived steady-state capital; Parameter values: $\beta = 0.99; \alpha = 1/3; \gamma = 2; \tau^* = 0.3; \sigma_\tau = 0.03; \sigma_\eta = 0.01$.

are chosen to be 5% lower, 3% lower, 1% lower, equals to, 1% higher or 5% higher than the actual tax target τ^*. The single tax shock hits the economy at $t = 1$(dotted red). Households update their beliefs at the end of each time period.[8] The dashed purple lines track the evolution of households' beliefs on tax targets, which gradually converge to the actual tax target τ^* given there are no subsequent shocks. Time-varying perceived tax targets directly affect households' perceived steady-state capital level(dash-dotted green). Instead of converging to zero directly, impulse responses of capital level(solid blue)

converge to the perceived steady-state capital level first. The two series then converge to 0 together as households' beliefs approaches to the correct tax target τ^*. Without technology shocks, households' beliefs $\tau^{\circledast}$'s influence both the perceived steady-state capital level $K_t^{\circledast}$ and the response function $f_1(\tau^{\circledast})$ in (9). These combined effects result the non-trivial impulse responses of capital.

In Figure 2 different households' initial beliefs $\tau_1^{\circledast}$'s generate a wide variety of impulse responses of capital. For example, in the bottom left panel of Figure 2, households' initial belief on tax target is 1% above its true value, which leads to lower perceived steady-state capital. A positive one standard deviation tax shock generate a realized tax rate 3% higher than its actual steady state and forces the households to further increase their perceived tax targets, even though they have already overestimated the steady-state tax rate by 1%. Higher perceived tax target further depresses households' saving, leading to a downward impulse response of capital during the first period. No further tax shocks indicate observed tax rates will remain at its steady-state level τ^* from $t = 2$. Even though households are still believe tax targets are time-varying($\sigma_\eta > 0$), observing a constant series of tax rates gradually pulls households beliefs back to the actual tax target τ^*. In the mean time, households adjust perceived steady-state capital level accordingly. The gradual adjustments can be seen from the behaviour of capital: Instead of converging to zero directly, impulse responses of capital converges to the perceived steady-state capital first.

In summary, households' beliefs on tax targets $\tau_t^{\circledast}$ play several crucial roles: it not only enters the economic system everywhere as the IIAU policy function (9) and the perceived steady state $K_t^{\circledast}$ imply, various beliefs $\tau_t^{\circledast}$'s also have the ability to generate *qualitatively* different impulse responses to the same tax shock. While these channels provide insights on how low-frequency

[8] In this simple growth model households can update their beliefs on tax targets as soon as current tax rate τ_t is realized, which could be at the beginning of each time period. In the following RBC model households can only update their beliefs at the end of each time period after realizations of some endogenous variables. To be consistent with the RBC model, I let households update beliefs at the end of each time period in the simple growth model. The updated belief will be utilized at the beginning of next period to form expectations. Changing the timing assumption to the beginning of each time period won't influence the results qualitatively.

fiscal uncertainty could impact the real economy, the simple analytical model does not shed light on whether they matter *quantitatively* in practice. In the following I tackle the households learning problem in a more realistic RBC framework.

3 A RBC model with a detailed fiscal section

I use a real model of a closed economy, augmented with fiscal details, to study the consequences of low-frequency fiscal uncertainty when households are learning unknown fiscal targets. The model structure follows Leeper et al. (2010). It is a RBC model extended to include inter- and intra- temporal preference shocks, investment adjustment costs, variable capacity utilization and external habit formation.[9] A detailed fiscal sector with distortionary taxes, government spending, lump-sum transfers, and one-period government bonds has been added on top of the model to analyze the impacts of fiscal financing on the real economy. These fiscal instruments are heterogeneous, each playing some unique role(s) of influencing the macro economy. They also respond to the state of economy endogenously. There are interactions between and among fiscal variables themselves. These features capture the fact that fiscal policies are complicated objects and ensure household's signal extraction problem on fiscal targets is non-trivial. The economy consists of a representative household, a representative firm, and a central government(the fiscal authority). The model structure $\mathcal{M}$ includes both the non-policy section $\mathcal{M}^{NP}$ and the fiscal policy section $\mathcal{M}^{FP}$.

3.1 The Model Structure $\mathcal{M}^{NP}$: Non-policy Section

3.1.1 Households

The representative household derives utility from consumption C_t, relative to a habit stock hC_{t-1}, where $h \in [0, 1]$. The household derives disutility from labor supply, L_t. There are two preference shocks, u_t^{β} and u_t^{ℓ}, that affect the household's intertemporal substitution between C_t's and intratemporal

[9] These real frictions do not affect household's learning problem and won't influence the underlying mechanisms how low-frequency fiscal uncertainty impact the economy *qualitatively*. However, variance decompositions provided in the online appendix of Leeper et al. (2010) show that they explain a large proportion of variations observed in aggregate data. To connect the model with data I preserve these real frictions.

substitution between C_t and L_t. The household maximize the intertemporal utility function

$$E_0^{\mathcal{I}} \sum_{t=0}^{\infty} \beta^t u_t^{\beta} \left[\frac{(C_t - hC_{t-1})^{1-\gamma}}{1-\gamma} - u_t^{\ell} \frac{L_t^{1+\kappa}}{1+\kappa} \right] \tag{12}$$

where $\gamma, \kappa > 0$ and $\beta \in (0,1)$. $E_t^{\mathcal{I}}$ is an expectation operator conditional on the household's information set $\mathcal{I}_t$(defined in section 4). The two preference shocks each follow a stationary AR(1) process

$$\log(u_t^{\beta}) = \rho_{\beta} \log(u_{t-1}^{\beta}) + \sigma_{\beta} \epsilon_t^{\beta}, \quad \epsilon_t^{\beta} \sim N(0,1);$$
$$\log(u_t^{\ell}) = \rho_{\ell} \log(u_{t-1}^{\ell}) + \sigma_{\ell} \epsilon_t^{\ell}, \quad \epsilon_t^{\ell} \sim N(0,1).$$

with $\rho_{\beta}, \rho_{\ell} \in [0,1)$. During each period household receive after-tax wage and capital rental income, lump-sum transfers Z_t from the government and spend income on consumption, investment I_t and government bonds B_t. The household's flow budget constraint is given by

$$(1+\tau_C^*)C_t + B_t + I_t = (1-\tau_t^L)W_t L_t + (1-\tau_t^K)R_t^K v_t K_{t-1} + R_{t-1}B_{t-1} + Z_t \tag{13}$$

where τ_C^*, τ_t^L, τ_t^K are tax rates on consumption C_t, labor income $W_t L_t$ and capital income $R_t^K v_t K_{t-1}$. The effective capital level supplied to firms is given by $v_t K_{t-1}$, where v_t measures capacity utilization in period t. R_t is the gross interest rate on the one-period, risk-free government bond B_t.

The law of motion for capital is given by

$$K_t = (1 - \delta(v_t))K_{t-1} + \left[1 - s(\frac{u_t^I I_t}{I_{t-1}}) \right] I_t \tag{14}$$

where following Christiano et al. (2005), $s(\frac{u_t^I I_t}{I_{t-1}})$ is an adjustment cost incurred if the household varies current investment from its previous level. $s(\cdot)$ satisfies $s(1) = s'(1) = 0$ and $s''(1) > 0$. The adjustment cost shock u_t^I follows

$$\log(u_t^I) = \rho_I \log(u_{t-1}^I) + \sigma_I \epsilon_t^I, \quad \epsilon_t^I \sim N(0,1).$$

Following Schmitt-Grohé and Uribe (2012), the capital utilization function δ follows a quadratic form,[10]

$$\delta(v_t) = \delta_0 + \delta_1(v_t - 1) + \frac{\delta_2}{2}(v_t - 1)^2. \tag{15}$$

[10]In (15) $\delta_1 = 1/\beta - (1 - \delta_0)$ so that the capacity utilization v_t equals 1 in the steady state.

The household maximizes utility (12), subject to the budget constraint (13), the law of motion for capital (14) and the quadratic variable utilization function (15).

3.1.2 Firms

The production function is Cobb-Douglas

$$Y_t = u_t^A (v_t K_{t-1})^\alpha L_t^{1-\alpha}. \tag{16}$$

where Y_t denotes the output produced with effective capital level $v_t K_{t-1}$, labor L_t and technology level u_t^A. Capital share $\alpha \in (0, 1)$. Technology follows a stationary AR(1) process

$$\log(u_t^A) = \rho_A \log(u_{t-1}^A) + \sigma_A \epsilon_t^A, \quad \epsilon_t^A \sim N(0, 1) \tag{17}$$

with $\rho_A \in [0, 1)$. The representative firm rents capital and labor from the household to maximize profit

$$Y_t - R_t^K v_t K_{t-1} - W_t L_t$$

subject to (16). Production market is competitive and wages W_t and capital rental rate R_t are paid at their marginal product

$$W_t = \frac{(1-\alpha) Y_t}{L_t},$$
$$R_t^K = \frac{\alpha Y_t}{v_t K_{t-1}}.$$

3.1.3 Government Budget Constraint

The government budget constraint is

$$B_t + \tau_t^K R_t^K v_t K_{t-1} + \tau_t^L W_t L_t + \tau_C^* C_t = R_{t-1} B_{t-1} + G_t + Z_t \tag{18}$$

where G_t is government spending. (18) imposes an intertemporal constraint on the government's fiscal instruments and is treated as an accounting identity. This concludes the non-policy section of the model structure $\mathcal{M}^{NP}$.

3.2 The Model structure $\mathcal{M}^{FP}$: Fiscal Policy

Fiscal policies are implemented in terms of simple rules. There is a lack of consensus on the "exact" form of these rules. However, typically three components are included: the first component represents "automatic stabilizers" fiscal policies usually designed to offset fluctuations in the macro economy. This is often modelled as responses of fiscal instruments to contemporaneous or lagged output. The second component allows fiscal instruments to adjust to the indebtedness of the government and is modelled as fiscal instruments' responses to contemporaneous/lagged debt gaps or debt-to-GDP ratios. This feature also guarantees the real debt is on a stable path and satisfy the transversality condition. The third component captures the fact that fiscal variables are highly persistent[11] and is modelled by including one or several AR(1) components in the fiscal rules. For recent estimated fiscal rules which include all three features, see Fernndez-Villaverde et al. (2015) and Leeper et al. (2015).

Fiscal rules considered here follow Leeper et al. (2010). Assuming a constant consumption tax rate τ_C^*, there are four simple rules: These rules govern the behaviour of government spending G_t, lump-sum transfers Z_t, capital and labor income tax rates τ_t^K, τ_t^L by allowing them to respond to the output and debt gaps to be defined below and to be autocorrelated. In terms of log deviations from steady states, the structural policy rules for $G_t, Z_t, \tau_t^K, \tau_t^L$ are

$$\begin{aligned}
\log(G_t) - \log(G^*) = & -\rho_{G,Y}[\log(Y_t) - \log(Y^*)] - \rho_{G,B}[\log(B_t) - \log(B^*)] \\
& + u_t^G, \quad u_t^G = \rho_G u_{t-1}^G + \sigma_G \epsilon_t^G; \qquad (19) \\
\log(Z_t) - \log(Z^*) = & -\rho_{Z,Y}[\log(Y_t) - \log(Y^*)] - \rho_{Z,B}[\log(B_t) - \log(B^*)] \\
& + u_t^Z, \quad u_t^Z = \rho_Z u_{t-1}^Z + \sigma_Z \epsilon_t^Z; \qquad (20) \\
\log(\tau_t^K) - \log(\tau_K^*) = & \rho_{K,Y}[\log(Y_t) - \log(Y^*)] + \rho_{K,B}[\log(B_t) - \log(B^*)] \\
& + u_t^K + \phi_{K,L} u_t^L, \quad u_t^K = \rho_K u_{t-1}^K + \sigma_K \epsilon_t^K; \qquad (21) \\
\log(\tau_t^L) - \log(\tau_L^*) = & \rho_{L,Y}[\log(Y_t) - \log(Y^*)] + \rho_{L,B}[\log(B_t) - \log(B^*)] \\
& + u_t^L + \phi_{K,L} u_t^K, \quad u_t^L = \rho_L u_{t-1}^L + \sigma_L \epsilon_t^L; \qquad (22)
\end{aligned}$$

[11] Along with many estimated DSGE models(e.g., Schmitt-Grohé and Uribe (2012); Smets and Wouters (2003);), my following estimated results reinforce the findings that the AR(1) autoregressive parameters for fiscal variables are close to 1.

Fiscal targets $G^*, Z^*, \tau_K^*, \tau_L^*$ and B^* are thus assumed to be constant. In consequence, along with the stationary technology process (17), there is a well-defined constant steady-state output level Y^*.[12] It follows $\log(Y_t) - \log(Y^*)$ defines the current output's deviation from steady state. The debt gap at time t is defined as $\log(B_t) - \log(B^*)$. In each fiscal rule (19), (20), (21) and (22) an AR(1) component has been included to allow autocorrelations of each fiscal instrument. Fiscal innovations $\{\epsilon_{G,t}, \epsilon_{Z,t}, \epsilon_{K,t}, \epsilon_{L,t}\}$ are $i.i.d\ N(0,1)$. Tax rates are correlated. Following Leeper et al. (2010), $\phi_{K,L}$ in (21), (22) captures how much unpredicted movement in one tax rate is due to an exogenous shock to another tax rate. Estimated results provides in the original paper and here show $\phi_{K,L}$ is significantly larger than 0 and improves model fit. Correlated tax rates also complicated household's signal extraction problem on unknown fiscal targets.

It is important to point out the fiscal authority's choices of fiscal targets are constrained. In particular, government cannot freely choose all 5 fiscal targets at once. Given time-invariant $G^*, Z^*, \tau_K^*, \tau_L^*$, Appendix B.3 shows these four fiscal targets pin down an unique steady-state debt level. Such steady-state debt level defines B^*. Equivalently, existence of such B^* can be derived by imposing the intertemporal equilibrium condition (IEC) to hold in steady state. Following Leeper (2010), government bond derives its value from anticipated fiscal backing, which comes from all tax revenues(i.e., consumption, capital and labor taxes revenues) net of total government outlays(i.e., government spending and lump-sum transfers). Imposing equilibrium conditions[13] on government budget constraint and taking conditional expectations give the IEC, which implies the real value of government debt must equal to the discounted present value of future primary surpluses. The steady-state IEC imposes an implicit restriction on fiscal targets: Given any four of $\{G^*, Z^*, \tau_K^*, \tau_L^*, B^*\}$, the fifth target must be compatible with the steady-state IEC.[14] The steady-state IEC links all fiscal targets and plays a crucial role in helping the households pin down their perceived fiscal targets in the following signal extraction problem.

[12] Appendix B.3 shows Y^* only depends on three fiscal targets: G^*, τ_K^* and τ_L^*.

[13] These equilibrium conditions include household's Euler equations, firm's first order conditions and the transversality conditions for debt and capital accumulation.

[14] There are other constraints fiscal targets naturally need to satisfy to be economically meaningful. For example, $0 \leq \tau_L^*, \tau_K^* < 1$ and $0 \leq G^*/Y^*, Z^*/Y^* < 1$. I assume all these constraints are satisfied.

Fiscal rules (19), (20), (21) and (22) constitute the *actual* fiscal policy $\mathcal{M}^{FP}$, which is different than the household's perceived fiscal policy $\mathcal{M}^{PFP}$(defined in section 4.1). $\{\mathcal{M}^{NP}; \mathcal{M}^{FP}\}$ define the complete model structure $\mathcal{M}$. $\mathcal{M}$ differs from Leeper et al. (2010)'s model in only one aspect: Instead of assuming a time-varying consumption tax rate τ_t^C, I let it be a constant τ_C^*. Estimation results in Leeper et al. (2010) reveal there's little variation of τ_t^C. Interactions between τ_t^C and other tax rates are also close to 0. At the same time, including a consumption tax rule won't complicate household's learning problem on steady-state consumption tax rate: Given a typical $i.i.d$ consumption tax rule with a time-invariant τ_C^*, households will quickly learn τ_C^*. By considering a constant consumption tax rate and assuming households have already learned τ_C^*, I shut down these minor channels completely. It follows the low-frequency fiscal uncertainty are concentrated on other fiscal targets: $G^*, Z^*, \tau_K^*, \tau_L^*$ and B^*.

3.3 Equilibrium conditions

In equilibrium households and firms are optimizing and the capital rental, labor and bond markets all clear. Debt and capital accumulation also satisfy the transversality conditions. The final goods market is in equilibrium if the aggregate demand by the household and government equals to the aggregate production:

$$Y_t = C_t + I_t + G_t \tag{23}$$

Appendix B.1 provides the complete equilibrium conditions of the RBC model.

4 Household's signal extraction problem

It remains to define the information set $\mathcal{I}_t$ in (12) before confronting the household's signal extraction problem on fiscal targets. Analogous to $\mathcal{I}_t^{full\ info.}$ defined in the simple analytical example, the FIRE information set, denoted by $\mathcal{I}_t^{FIRE}$, is given by

$$\mathcal{I}_t^{FIRE} = \{\mathcal{M}^{NP}; \mathcal{M}^{FP}; \mathcal{O}_t; \mathcal{P}\}$$

where the model structure consists of $\{\mathcal{M}^{NP}; \mathcal{M}^{FP}\}$, $\mathcal{O}_t$ includes the entire history of observables and $\mathcal{P}$ contains all parameters of the model. For a detailed description of $\mathcal{O}_t$ and $\mathcal{P}$, see appendix B.2. It is important to emphasize the AR(1) components of fiscal policies u_t^G, u_t^Z, u_t^K and u_t^L are not directly observable to the households and are not in $\mathcal{O}_t$. However, knowledge of $\mathcal{M}^{FP}, \mathcal{O}_t$ and all parameters $\mathcal{P}$ guarantee households can easily derive these AR(1) components perfectly.[15] The ability to infer these fiscal AR(1) components changes substantially in the following incomplete information case. History of shocks are not directly observable to households and are not explicitly given in $\mathcal{I}_t^{FIRE}$. However, since households can figure out all stationary AR(1) components under $\mathcal{I}_t^{FIRE}$, one implication is that the history of all shocks are also contained in $\mathcal{I}_t^{FIRE}$.

Let the representative household's information set for the incomplete information, anticipated utility(IIAU) model be denoted by $\mathcal{I}_t^{IIAU}$. $\mathcal{I}_t^{IIAU}$ and $\mathcal{I}_t^{FIRE}$ differs in several aspects. To focus on policy uncertainties, I assume households completely understand the non-policy model structure $\mathcal{M}^{NP}$. Let the household's perceived fiscal policy be denoted by $\mathcal{M}^{PFP}$(defined in section 4.1), differences between *perceived* policy $\mathcal{M}^{PFP}$ and *actual* policy $\mathcal{M}^{FP}$ define and drive the consequences of low-frequency fiscal uncertainty. Let $\mathcal{P}_1$ be the parameter list in $\mathcal{I}_t^{IIAU}$. Since households are learning fiscal targets under IIAU, it is natural to exclude $\{G^*, Z^*, \tau_K^*, \tau_L^*, B^*\}$ from $\mathcal{P}_1$, i.e.,

$$\mathcal{P}_1 = \mathcal{P} \setminus \{G^*, Z^*, \tau_K^*, \tau_L^*, B^*\}.$$

$\mathcal{P}_1$ implies households have learned all other parameters in the RBC model, including all response coefficients $\phi_{K,L}$, $\rho_{X,Y}, \rho_{X,B}, \rho_X$ and standard deviations σ_X for $X = \{G, X, K, L\}$ in rules (19), (20), (21) and (22). The $\mathcal{I}_t^{IIAU}$ is given by

$$\mathcal{I}_t^{IIAU} = \{\mathcal{M}^{NP}; \mathcal{M}^{PFP}; \mathcal{O}_t; \mathcal{P}_1\}.$$

4.1 The household's perceived fiscal policy $\mathcal{M}^{PFP}$

Contrary to the *structural* fiscal rules defined in $\mathcal{M}^{FP}$, households learn *reduced-form* rules. They observe realized fiscal instruments and act as econometricians to infer the structure of fiscal policy. Their perceived fiscal policy

[15] To derive Y^* in $\mathcal{M}^{FP}$, households also need to utilize the non-policy model structure $\mathcal{M}^{NP}$.

$\mathcal{M}^{PFP}$ is given by

$$\begin{aligned}
\log(G_t) &= \Gamma_{G,t} - \rho_{G,Y}\log(Y_t) - \rho_{G,B}\log(B_t) + u_t^G, && (24)\\
\log(Z_t) &= \Gamma_{Z,t} - \rho_{Z,Y}\log(Y_t) - \rho_{Z,B}\log(B_t) + u_t^Z, && (25)\\
\log(\tau_t^K) &= \Gamma_{K,t} + \rho_{K,Y}\log(Y_t) + \rho_{K,B}\log(B_t) + u_t^K + \phi_{K,L}u_t^L, && (26)\\
\log(\tau_t^L) &= \Gamma_{L,t} + \rho_{L,Y}\log(Y_t) + \rho_{L,B}\log(B_t) + u_t^L + \phi_{K,L}u_t^K && (27)
\end{aligned}$$

where for $X = \{G, Z, K, L\}$, $u_t^X = \rho_X u_{t-1}^X + \sigma_X \epsilon_t^X$ are the AR(1) components in the fiscal rules. The above $\mathcal{M}^{PFP}$ indicates households have learned most aspects of fiscal policy: Not only have they successfully learned all response coefficients $\{\rho_{X,Y}, \rho_{X,B}, \rho_X, \phi_{K,L}\}$ in fiscal rules, they also figure out how these parameters enter fiscal rules correctly. The standard deviations σ_X's and structures of the additive AR(1) components are also in the households' information set. The low-frequency uncertainties thus have been isolated from uncertainties regarding other parameters in the fiscal rules.

Households are still learning intercept terms in $\mathcal{M}^{PFP}$ and the t subscripts attached to $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}$ and $\Gamma_{L,t}$ identify estimates of that coefficient at time t. The representative household's signal extraction problem is to keep track of $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}$ and $\Gamma_{L,t}$ first, then pin down the underlying unknown fiscal targets by combining his knowledge of the non-policy model structure $\mathcal{M}^{NP}$. Appendix B.3 gives a set of equations the actual steady states need to satisfy. Appendix B.4 provides a mapping between the household's perceived steady states of the economy and $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}$ and $\Gamma_{L,t}$.

4.2 Evolution of household's beliefs

Households act as Bayesian econometricians and update their beliefs on $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}$ by Kalman filters. All households share the same beliefs. They utilize their own knowledge of $\mathcal{M}^{PFP}$ and establish a linear state-space model of which a standard Kalman filter can be applied. Since households don't observe $\{u_t^G, u_t^Z, u_t^K, u_t^L\}$ directly, they also need to update their beliefs on these AR(1) components. Define

$$\xi_t = \{\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}, u_{G,t}, u_{Z,t}, u_{K,t}, u_{L,t}\},$$

ξ_t defines state variables in the household's learning problem. Households have already learned the law of motion for $\{u_{G,t}, u_{Z,t}, u_{K,t}, u_{L,t}\}$ in ξ_t: They

each follow an AR(1) process, as defined in the actual policy $\mathcal{M}^{FP}$.

Since $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}$ and $\Gamma_{L,t}$ in $\mathcal{M}^{PFP}$ encode sufficient information to determine household's perceived fiscal targets, the perceived law of motion for $\Gamma_{X,t}$ plays a crucial role: How do households think $\Gamma_{X,t}$ change over time for $X = \{G, Z, K, L\}$? Following Hollmayr and Matthes (2015), I endow households with a perceived law of motion that has been the benchmark on time-varying coefficient models in empirical macroeconomics [16]

$$\Gamma_{X,t} = \Gamma_{X,t-1} + \eta_{X,t}. \tag{28}$$

Direct comparisons of $\mathcal{M}^{FP}$ and $\mathcal{M}^{PFP}$ indicate the intercept terms in the actual fiscal policy $\mathcal{M}^{FP}$ are all constant. In particular, let $\{\Gamma^*_G, \Gamma^*_Z, \Gamma^*_K, \Gamma^*_L\}$ be the constant terms in the actual reduced-form $\mathcal{M}^{FP}$,

$$\begin{aligned}
\Gamma^*_G &= \log(G^*) + \rho_{G,Y}\log(Y^*) + \rho_{G,B}\log(B^*) \\
\Gamma^*_Z &= \log(Z^*) + \rho_{Z,Y}\log(Y^*) + \rho_{Z,B}\log(B^*) \\
\Gamma^*_K &= \log(\tau^*_K) - \rho_{K,Y}\log(Y^*) - \rho_{K,B}\log(B^*) \\
\Gamma^*_L &= \log(\tau^*_L) - \rho_{L,Y}\log(Y^*) - \rho_{L,B}\log(B^*)
\end{aligned}$$

Households beliefs on fiscal targets are correct if and only if $\Gamma_{G,t} \equiv \Gamma^*_G, \Gamma_{Z,t} \equiv \Gamma^*_Z, \Gamma_{K,t} \equiv \Gamma^*_K, \Gamma_{L,t} \equiv \Gamma^*_L$ for all t. It follows $\eta_{X,t} \equiv 0$ for $X = \{G, Z, K, L\}$. Define $\eta_t = [\eta_{G,t}, \eta_{Z,t}, \eta_{K,t}, \eta_{L,t}]'$, the correct perceived law of motion implies a zero covariance matrix of η_t.

As in the simple analytical model where households think the single tax target τ^* is time-varying, I now consider a perceived law of motion in which households think $\Gamma_{X,t}$ will be time-varying. There are mainly three reasons.

Firstly, since there are no explicitly announced $G^*, Z^*, \tau^*_K, \tau^*_L$ and B^*, households' beliefs on fiscal targets are not anchored. In consequence, there is no reason to prevent households from suspecting fiscal targets may be time-varying *per se*. If households believe fiscal targets may be changing, then it follows $\Gamma_{X,t}$'s are time-varying too.

[16] Hollmayr and Matthes (2015) considers a one-time policy change and allow households to know the exact timing of policy change. This assumption is reflected by putting an indicator function $\mathbf{1}_t$ in front of $\eta_{X,t}$. Here I assume households always suspect fiscal targets may be time-varying.

Secondly, estimation results show that all AR(1) components in $\mathcal{M}^{FP}$ are rather persistent, with autoregressive parameters all being extremely close to unity. It is notoriously difficult to distinguish a highly persistent but stationary process from an unit root process, especially in finite samples. Such close to unit root specification matches the observed slow-moving fiscal behaviour well, but raises a question on how households inside the economy would perceive an observed change in one fiscal variable: is it permanent or temporary? Given their beliefs on fiscal targets are not anchored, if households interpret an observed change in labor tax rates as permanent, then they shift their beliefs on steady state τ_L^*, which in consequence will change their belief on $\Gamma_{L,t}$.

Thirdly, as argued in the introduction, no one knows the fiscal targets whether they are time-varying or time-invariant. To take the model to data economists and econometricians however need to take a stance on *assuming* the true data generating process and on properties of fiscal targets in particular. Acting as both the economist and the econometrician, I make the simplest *assumption* that fiscal targets $\{G^*, Z^*, \tau_K^*, \tau_L^*, B^*\}$ considered here are all time-invariant. By allowing household to suspect fiscal targets may be time-varying thus creates a mismatch between households' beliefs and the true data generating process. Such mismatch originates from the fact that households do not know fiscal targets in the first place and is reinforced by the empirical findings that fiscal variables are highly persistent. It is not due to the assumption that fiscal targets are time-invariant. In fact, more structure could be put into the model to generate actual time-varying fiscal targets. As long as households' perceived fiscal targets are different than the actual ones, such mismatch still exists.

Let the covariance matrix of η_t be denoted by Σ_η. I set Σ_η to be a diagonal matrix. Along with (28), such Σ_η has a natural behaviour interpretation: Households believe there are some underlying "shocks" $\eta_{X,t}$ to $\Gamma_{X,t}$ and such "shocks" represent the fiscal authority's discretionary power to change the intercept terms in the fiscal rules. I further assume households' perceived "shocks" $\eta_{X,t}$ are mutually independent and are independent of $\epsilon_{X,t}$: Households thus believe fiscal authority's discretionary power to change the intercept term in one fiscal rule is independent of changes of the intercept terms in other fiscal rules. Only when Σ_η equals to the 4×4 zero matrix households beliefs are correctly specified.

Appendix B.5 establishes households' learning problem as the following linear state-space model

$$\begin{aligned} \xi_{t+1} &= F\xi_t + v_{t+1} \\ \Omega_t &= H'\xi_t + w_t \\ E(v_t v_\tau') &= \begin{cases} Q & \text{for } t = \tau \\ 0 & \text{otherwise} \end{cases} \\ E(w_t w_\tau') &= R \equiv 0, \quad \forall t, \tau; \end{aligned} \tag{29}$$

where Ω_t's are the observables in the state-space model and include linear combinations of realized fiscal instruments and output(in logs). Q is the covariance matrix of v_t, where v_t is an 8×1 vector including households' perceived shocks $\eta_{X,t}$ and the underlying fiscal innovations $\sigma_X \epsilon_{X,t}$ for $X = \{G, Z, K, L\}$,

$$v_t = [\eta_{G,t}, \eta_{Z,t}, \eta_{K,t}, \eta_{L,t}, \sigma_G \epsilon_{G,t}, \sigma_Z \epsilon_{Z,t}, \sigma_K \epsilon_{K,t}, \sigma_L \epsilon_{L,t}]'.$$

The covariance matrix Q is assumed to be time-invariant. Appendix B.5 shows Q is a 8×8 diagonal matrix and is of the form

$$Q = \begin{pmatrix} \sigma^2_{\eta,G} & & & & & & & \\ & \sigma^2_{\eta,Z} & & & & & & \\ & & \sigma^2_{\eta,K} & & & & & \\ & & & \sigma^2_{\eta,L} & & & & \\ & & & & \sigma^2_G & & & \\ & & & & & \sigma^2_Z & & \\ & & & & & & \sigma^2_K & \\ & & & & & & & \sigma^2_L \end{pmatrix} \tag{30}$$

The following well known Kalman recursions[17] characterize the evolution of households' beliefs:

$$\xi_{t+1|t} = F\xi_{t|t-1} + FP_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}(\Omega_t - H'\xi_{t|t-1}), \tag{31}$$

$$P_{t+1|t} = F[P_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}]F' + Q; \tag{32}$$

[17] See equations (13.2.18) and (13.2.22) of Hamilton (1994).

where $\xi_{t+1|t}$ contains the one period ahead forecast of the policy rule coefficients $\Gamma_{X,t}$ and the one period ahead forecast of AR(1) components $u_{X,t}$ produced by the Kalman filter, $P_{t+1|t}$ is a 8×8 positive definite matrix which characterizes households' confidence level with respect to their forecast $\xi_{t+1|t}$. The pair $(\xi_{t+1|t}, P_{t+1|t})$ consists of a set of sufficient statistics that summarizes the households' beliefs to the unknown fiscal targets after observing the history of Ω_t.

4.3 Solving the RBC model

The solution procedure follows Cogley et al. (2015) and Hollmayr and Matthes (2015) and consists of three stages. In any period $t > 0$ households first solve their perceived model given beliefs $(\xi_{t|t-1}, P_{t|t-1})$ and derive their perceived law of motion of endogenous variables. The perceived law of motion then drives the actual law of motion and the equilibrium dynamics of the economy. After observing the realization of fiscal instruments and output, households update their beliefs by running the Kalman filter and form $(\xi_{t+1|t}, P_{t+1|t})$. The RBC model can then be solved recursively. Appendix B.6 provides details of the algorithm.

5 Empirical Results

In this section I take the model to data. I first estimate deep structural parameters of the models by the Bayesian approach. I then simulate both FIRE and IIAU models by plotting impulse response and comparing differences. Finally I extract household's recent beliefs on fiscal targets using a Rao-Blackwellized particle filter(RBPF).

I estimate the FIRE model using the Kalman filter: Log-linearizing the equilibrium conditions around the deterministic steady states and solving the resulted log-linearized model[18] yield a linear state transition equation. Combined with a linear measurement equation, the FIRE model parameters can be estimated using the Kalman filter.

Kalman filters, however, cannot be applied directly to the above IIAU model to estimate parameters. This is because households' beliefs $(\xi_{t|t-1}, P_{t|t-1})$

[18] I solve the log-linearized model using Sims (2002) Gensys algorithm.

act as additional state variables and enter the model system nonlinearly. Direct parameter estimation (either the frequentist approach or the Bayesian approach) of the non-linear state-space model by sequential Monte Carlo methods is computationally intensive. However, a closer look at the model algorithm and solution(See Appendix B.6) indicates most of the nonlinearities households' beliefs bringing into the model comes from their estimates of $\{\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}\}$, which is the first four elements of $\xi_{t|t-1}$. The other part of the beliefs, the covariance matrix $P_{t|t-1}$ which governs households' confidence level of $\xi_{t|t-1}$, only enters the system nonlinearly through (31). What's more, the second Kalman recursion equation (32) indicates the evolution of $P_{t|t-1}$ is completely autonomous, i.e., $P_{t+1|t}$ only depends on $P_{t|t-1}$, and $P_{t-1|t}$ only depends on its previous counterpart $P_{t-2|t-1}$. It follows $P_{t|t-1}$ for all t are completely determined by $P_{0|-1}$, which is the confidence level of households to their initial beliefs $\xi_{0|-1}$. To downplay the roles of initial beliefs, I consider a steady-state Kalman filter with $P_{0|-1} = P$, where P solves the discrete-time algebraic Riccati equation

$$P = F[P - PH(H'PH + R)^{-1}H'P]F' + Q. \tag{33}$$

I then log-linearize the IIAU model around the actual steady states of the economy by recognizing an expanded set of state variables. For a given set of parameters, the log-linearized IIAU model only requires to evaluate a Jacobian term by solving the full non-linear IIAU model twice. Appendix C.1 establishes a linear state transition equation for the linearized IIAU model. The following sections estimate both the FIRE and the linearized IIAU model.

5.1 Estimation

Structural parameters are estimated with US quarterly data from 1959Q3 to 2016Q1. Data consists of eight time series: consumption, investment, hour worked, government spending, lump-sum transfers, government debt, capital and labor tax revenues.[19] Appendix C.2 describes the data source and construction. In contrast to Leeper et al. (2010) who detrends the logarithm of each observable independently with a linear trend, I only linearly detrend consumption, investment and hours worked. The remaining fiscal

[19]Since quarterly measure of *marginal* tax rates are problematic, I avoid using tax rates as observables. Following Jones (2002), average capital and labor tax rates are calculated first, tax revenues are then constructed by multiplying average tax rates with tax bases.

observables are not detrended and variable-to-GDP ratios are constructed to preserve low-frequency movements of fiscal instruments. Appendix C.3 establishes the measurement equation which links observables to model variables.

It is important to preserve low-frequency movements of fiscal variables by avoid detrending them. Firstly, the seemingly upward or downward "trends"[20] of fiscal variables contain useful information on households' perceived fiscal targets. Households may shift their beliefs on steady-state fiscal variables after observing an increasing fiscal instrument for a long time. It is also the source why households may think $\Gamma_{X,t}$'s are time-varying in (28) in the first place. Secondly, not detrending fiscal variables yields very close to unity estimates of autoregressive parameters in fiscal rules, giving households a difficult signal extraction problem (29) to disentangle $\Gamma_{X,t}$ from $u_{X,t}$. The former series, $\Gamma_{X,t}$, are perceived as random walks while the later series $u_{X,t}$'s will display near random walk behaviour due to rather persistent AR(1) coefficients.

The FIRE model contains parameters $\mathcal{P}$ which also appear in the IIAU model. The IIAU model contains four additional structural parameters $\sigma_{\eta,G}$, $\sigma_{\eta,Z}$, $\sigma_{\eta,K}$, and $\sigma_{\eta,L}$. These parameters characterize how time-varying households think $\Gamma_{X,t}$'s are. I estimate both the FIRE model and the linearized IIAU model using Bayesian methods to recover $\mathcal{P}$ and $\{\sigma_{\eta,G}, \sigma_{\eta,Z}, \sigma_{\eta,K}, \sigma_{\eta,L}\}$. The parameters' posterior distributions, which is a combination of priors and the likelihood function, are calculated using Kalman filter. I use the random walk Metropolis-Hastings(MH) algorithm to sample from the posterior distributions. The proposal density used in the MH algorithm is found by using Sims optimization routine csminwel to maximize the log posterior function.[21] For each model specification a total of 1.5 million draws was created, with the first 500,000 draws discarded as a burn-in process. Every 50th draws was kept to remove correlation of the draws. The sample size from the posterior equals to 20,000.

[20]The word "trends" are in parenthesis since they may not be trends and cannot last forever. Fiscal variables are highly persistent and thus can display very long cycles.

[21]Step sizes are tuned in the MH algorithms to target acceptance ratios in the range of 20 to 40 percent. I use trace plots, ACF plots and Geweke's separated partial means tests to diagnose the chain convergence.

Parameter	Value
β	0.99
δ_0	0.025
α	0.33
G^*/Y^*	0.0922
B^*/Y^*	0.3396
τ_K^*	0.184
τ_L^*	0.223
τ_C^*	0.0287

Table 1: Calibrated parameters

In both FIRE and IIAU model specifications I fix several parameters. The quarterly discount factor β is set to 0.99. The quarterly depreciation rate for capital, δ_0, is set to 0.025, implying a 10 percent annual depreciation rate. The capital income share of output, α, is set to 0.3. Steady-state tax rates and ratios of government spending and debt to output are taken from Leeper et al. (2010). These values are calibrated to match the historical means of data from 1960Q1 to 2008Q1. Table 1 lists all calibrated parameter values. Notice that only four fiscal targets $G^*/Y^*, B^*/Y^*, \tau_K^*$ and τ_L^* are calibrated, the steady-state IEC implied by the model structure $\mathcal{M}$ imposes a restriction on fiscal targets and will pin down the fifth fiscal target Z^*.[22]

Tables 2 and 3 list prior distributions for the remaining parameters. Parameters are assumed independent *a priori*. Prior distributions follow Leeper et al. (2010) with a few exceptions. These priors cover a broad range of parameter values and are similar to those commonly used in the literature (See Smets and Wouters (2007); An and Schorfheide (2007); and Forni et al. (2009)). Table 2 and 3 also report means and standard errors implied by these prior distributions. The means are set at values that correspond with estimates of related studies in the literature. Standard errors are chosen to cover a large range of parameter values. Response coefficients in fiscal rules play important roles in governing how fiscal variables respond to output and finance debt. These priors are chosen to ensure the domains cover previous estimated results. Leeper et al. (2010) uses relative restrictive priors on $\rho_{G,Y}$

[22] Or equivalently, Z^*/Y^* since the model environment is stationary

Parameter		**Prior**	
	distribution	mean	std.
Preference and HHs			
h, habit formation	Beta	0.50	0.20
γ, risk aversion	Gamma	1.75	0.50
κ, inverse Frisch elast.	Gamma	2.00	0.50
s'', investment adj. cost	Gamma	5.00	0.25
δ_2, capital util. cost	Gamma	0.70	0.50
AR(1) coefficients			
ρ_β, β pref. AR coeff.	Beta	0.70	0.20
ρ_ℓ, labor pref. AR coeff.	Beta	0.70	0.20
ρ_I, inv. AR coeff.	Beta	0.70	0.20
ρ_A, tech. AR coeff.	Beta	0.70	0.20
ρ_G, gov. spend. AR coeff.	Beta	0.70	0.20
ρ_Z, transfer AR coeff.	Beta	0.70	0.20
ρ_K, cap. tax AR coeff.	Beta	0.70	0.20
ρ_L, labor tax AR coeff.	Beta	0.70	0.20
Response coefficients in fiscal rules			
$\phi_{K,L}$, cap./labor co-term	Normal	0.20	0.10
$\rho_{G,Y}$, gov. spend output coeff.	Gamma	0.20	0.30
$\rho_{Z,Y}$, transfer output coeff.	Gamma	0.20	0.30
$\rho_{K,Y}$, cap. tax output coeff.	Gamma	0.20	0.30
$\rho_{L,Y}$, labor tax output coeff.	Gamma	0.20	0.30
$\rho_{G,B}$, gov. spend debt coeff.	Gamma	0.40	0.20
$\rho_{Z,B}$, transfer debt coeff.	Gamma	0.40	0.20
$\rho_{K,B}$, cap. tax debt coeff.	Gamma	0.40	0.20
$\rho_{L,B}$, labor tax debt coeff.	Gamma	0.40	0.20

Table 2: Prior distributions

Parameter	Prior		
	distribution	mean	std.
Std. dev. of innovations			
σ_β, β pref. std.	Inv. Gamma	1	4
σ_ℓ, labor pref. std.	Inv. Gamma	1	4
σ_I, inv. std.	Inv. Gamma	1	4
σ_A, tech. std.	Inv. Gamma	1	4
σ_G, gov. spend. std.	Inv. Gamma	1	4
σ_Z, transfer std.	Inv. Gamma	1	4
σ_K, cap. tax std.	Inv. Gamma	1	4
σ_L, labor tax std.	Inv. Gamma	1	4
Std. dev. of perceived "shock"			
$100\sigma_{\eta,G}$, $\eta_{G,t}$ std.	Gamma	0.4	0.2
$100\sigma_{\eta,Z}$, $\eta_{Z,t}$ std.	Gamma	0.4	0.2
$100\sigma_{\eta,K}$, $\eta_{K,t}$ std.	Gamma	0.4	0.2
$100\sigma_{\eta,L}$, $\eta_{L,t}$ std.	Gamma	0.4	0.2

Table 3: Prior distributions [continued]

and $\rho_{Z,Y}$. Not detrending fiscal variables seems to suggest data prefer larger elasticities of fiscal instruments with respect to output. I thus use a common Gamma distribution with a mean of 0.2 and standard deviation of 0.3 as priors for fiscal responses to output(the $\rho_{X,Y}$'s). The standard deviations of perceived shocks $\{\sigma_{\eta,G}, \sigma_{\eta,Z}, \sigma_{\eta,K}, \sigma_{\eta,L}\}$ in (28) are not available in Leeper et al. (2010) and a Gamma distribution with a mean of 0.4 and standard deviation of 0.2 is chosen as priors. These priors are fairly diffuse and cover a large range approximately between 0 and 1.25.

Tables 4 and 5 present the means and 5% and 95% of the posterior distribution for the model specifications estimated, along with Leeper et al. (2010)'s posterior means and 90% credible intervals. Figure 3 and 4 visualize these results by comparing prior and posterior distributions of various model specifications.

5.1.1 Comparing FIRE and Leeper et al. (2010) posteriors

I first compare the differences between FIRE and Leeper et al. (2010)'s posteriors. These differences are mainly due to different detrending methods.[23] For non-policy parameters, results are roughly the same and are similar to estimates in the literature . My estimate of the external habit stock is about 70% of past consumption, which is significantly larger than Leeper et al. (2010)'s mean estimates, 0.5, but is in line with many other macro models' estimates with external habits(See Havranek et al. (2015) for a meta study reports). Compared with Leeper et al. (2010), the data now prefers a larger σ_β(10.2 vs 7) and a smaller ρ_β(0.43 vs. 0.66) of the preference shock on discount factor. Larger demand shocks and stronger habit formation thus are required to fit data well once fiscal variables are not detrended.

Turning to policy parameter estimates, differences are larger and several patterns arise. First of all, not detrending fiscal variables preserve the strong persistence of instruments observed in data. In consequence, my estimated AR(1) coefficients in fiscal rules are consistently larger than Leeper et al. (2010)'s and are all very close to unity. Three autoregressive coefficients ρ_G, ρ_Z, ρ_L's 90% credible intervals are concentrated around 0.995 with very small standard deviations(0.0032, 0.0036, 0.0034). The autoregressive parameter for the capital tax rule, ρ_K, has a slightly smaller posterior mean, 0.976, and a wider 90% credible interval [0.96, 0.99]. These estimates are in line with Fernndez-Villaverde et al. (2015)'s results, who also do not detrend fiscal variables and use variable-to-GDP ratios as observables.

AR(1) parameters close to unity play a crucial role in households' signal extraction problem (29). With near unit root behaviour, it is plausible to suspect, at one time or another, households may perceive movements of certain fiscal variable are due to changes to the underlying fiscal targets,

[23] I also estimate the FIRE models with both Pre-Great Recession data(1960Q1-2008Q1) and full available data (1959Q3-2016Q1); with/without adding independent measurement errors to observables; Posterior distributions of parameters are robust with respect to these specifications as long as fiscal variables are not detrended and variable-to-GDP ratios are used as observables. Standard deviations of measurement errors are small and well-identified. Detrending each fiscal variable linearly and re-estimating the FIRE models recover most of Leeper et al. (2010)'s estimation results, either by using the 1960Q1-2008Q1 data or the full available 1959Q3 -2016Q1 data, with/without adding measurement errors.

especially when their beliefs on fiscal targets are not anchored. Near unit root behaviour also makes the fiscal authority hard to anchor households' expectations forever. The fiscal authority may credibly announce its fiscal targets[24] and anchor households' expectations temporarily. In the future households can *always* suspect fiscal targets may have changed as long as there are low-frequency, persistent movements of fiscal variables deviating them from their announced targets. Since fiscal variables are highly endogenous and are subject to all sorts of exogenous shocks hitting the economy, households' signal extraction problem on fiscal targets is non-trivial.

More diffuse priors along with different detrending method yield larger fiscal instruments' elasticities with respect to output (the $\rho_{X,Y}$'s). In particular, my estimated $\rho_{G,Y}$ and $\rho_{Z,Y}$ are significantly larger, indicating stronger responses of fiscal spending to output. Government spending, lump-sum transfers and capital tax's elasticities with respect to debt(i.e., $\rho_{G,B}$, $\rho_{Z,B}$ and $\rho_{K,B}$) matches Leeper et al. (2010)'s estimates well. Elasticity of labor tax with respect to debt is much larger: The mean estimate of $\rho_{L,B}$ indicates a 1% increase in debt leads to a 0.5% increase in labor tax rates. Different estimates of fiscal response coefficients will have large impacts on related fiscal calculations such as multipliers and policy evaluations(Mountford and Uhlig (2009)).

Finally, not detrending fiscal variables yields slightly larger estimates of fiscal innovations' standard deviations(the σ_X's) across government spending, lump-sum transfers, capital and labor taxes. This result, along with larger AR(1) coefficients estimates, indicates data now prefer bigger fiscal shocks to absorb variations embedded in the low-frequency movements of fiscal variables.

[24]Due to the steady-state IEC constraint, fiscal targets are constrained and the fiscal authority cannot target all targets independently at the same time.

Parameter	Leeper et al. (2010)		FIRE		Linearized IIAU	
	mean	90% C.S.	mean	90% C.S.	mean	90% C.S.
Preference and HHs						
h, habit formation	0.50	[0.4, 0.6]	0.70	[0.59, 0.79]	0.71	[0.60, 0.81]
γ, risk aversion	2.7	[2.1, 3.4]	2.33	[1.57, 3.19]	2.14	[1.39, 3.00]
κ, inverse Frisch elast.	1.9	[1.4, 2.6]	2.98	[2.08, 4.06]	2.87	[2.04, 3.89]
s'', investment adj. cost	5.5	[5.1, 5.9]	5.72	[5.29, 6.15]	5.70	[5.27, 6.13]
δ_2, capital util. cost	0.29	[0.2, 0.42]	0.80	[0.47, 1.34]	0.82	[0.48, 1.36]
AR(1) coefficients						
ρ_β, β pref. AR coeff.	0.66	[0.62,0.69]	0.44	[0.38, 0.50]	0.43	[0.37, 0.49]
ρ_ℓ, labor pref. AR coeff.	0.99	[0.97,0.99]	0.98	[0.97, 0.99]	0.98	[0.97, 0.99]
ρ_I, inv. AR coeff.	0.55	[0.47,0.64]	0.49	[0.39,0.59]	0.47	[0.36, 0.57]
ρ_A, tech. AR coeff.	0.96	[0.94, 0.98]	0.99	[0.98,0.99]	0.99	[0.98, 0.99]
Std. dev. of inno.						
$100\sigma_\beta$, β pref. std.	7	[6.4, 7.7]	10.58	[9.65,11.60]	10.20	[9.24, 11.24]
$100\sigma_\ell$, labor pref. std.	2.8	[2.3, 3.5]	4.21	[3.22,5.44]	4.07	[3.14, 5.21]
$100\sigma_I$, inv. std.	6.4	[5.7, 7.2]	6.01	[5.27, 6.88]	5.79	[5.03, 6.64]
$100\sigma_A$, tech. std.	0.62	[0.58, 0.68]	0.62	[0.57,0.67]	0.62	[0.57, 0.67]

Table 4: Non-policy parameter posterior distributions: means and 90-percent credible sets.

Parameter	Leeper et al. (2010)		FIRE		Linearized IIAU	
	mean	90% C.S.	mean	90% C.S.	mean	90% C.S
AR(1) coefficients						
ρ_G, gov. spend. AR coeff.	0.97	[0.95, 0.99]	0.99	[0.99, 0.99]	0.99	[0.99, 0.99]
ρ_Z, transfer AR coeff.	0.94	[0.91, 0.98]	0.99	[0.99, 0.99]	0.99	[0.99, 0.99]
ρ_K, cap. tax AR coeff.	0.93	[0.9, 0.97]	0.97	[0.96, 0.99]	0.96	[0.94, 0.98]
ρ_L, labor tax AR coeff.	0.97	[0.95, 0.99]	0.99	[0.99, 0.99]	0.99	[0.99, 0.99]
Response coeff. in $\mathcal{M}^{FP}$						
$\phi_{K,L}$, cap./labor co-term	0.19	[0.14, 0.24]	0.22	[0.17,0.27]	0.23	[0.18, 0.28]
$\rho_{G,Y}$, gov. spend output coeff.	0.034	[0.0064, 0.084]	0.73	[0.43,1.08]	0.68	[0.40, 1.01]
$\rho_{Z,Y}$, transfer output coeff.	0.13	[0.049, 0.24]	1.03	[0.65,1.45]	1.02	[0.65, 1.44]
$\rho_{K,Y}$, cap. tax output coeff.	1.7	[1.2, 2.1]	2.47	[1.90,3.05]	2.45	[1.87, 3.04]
$\rho_{L,Y}$, labor tax output coeff.	0.36	[0.16, 0.61]	0.63	[0.38, 0.92]	0.66	[0.39, 0.97]
$\rho_{G,B}$, gov. spend debt coeff.	0.23	[0.15, 0.31]	0.13	[0.06, 0.22]	0.12	[0.05, 0.20]
$\rho_{Z,B}$, transfer debt coeff.	0.5	[0.41, 0.59]	0.39	[0.26, 0.54]	0.39	[0.25, 0.53]
$\rho_{K,B}$, cap. tax debt coeff.	0.39	[0.28, 0.51]	0.41	[0.26,0.57]	0.43	[0.28, 0.59]
$\rho_{L,B}$, labor tax debt coeff.	0.049	[0.019, 0.09]	0.45	[0.33, 0.57]	0.47	[0.35, 0.60]
Std. dev. of inno.						
$100\sigma_G$, gov. spend. std.	3.1	[2.8, 3.3]	3.55	[3.26, 3.87]	3.51	[3.23, 3.81]
$100\sigma_Z$, transfer std.	3.4	[3.1, 3.7]	4.03	[3.69, 4.39]	4.01	[3.68, 4.38]
$100\sigma_K$, cap. tax std.	4.4	[4.1, 4.7]	4.58	[4.20, 4.99]	4.60	[4.22, 5.01]
$100\sigma_L$, labor tax std.	3	[2.8, 3.2]	3.04	[2.78, 3.34]	3.09	[2.81, 3.39]

Table 5: Policy parameters posterior distributions: means and 90-percent credible sets.

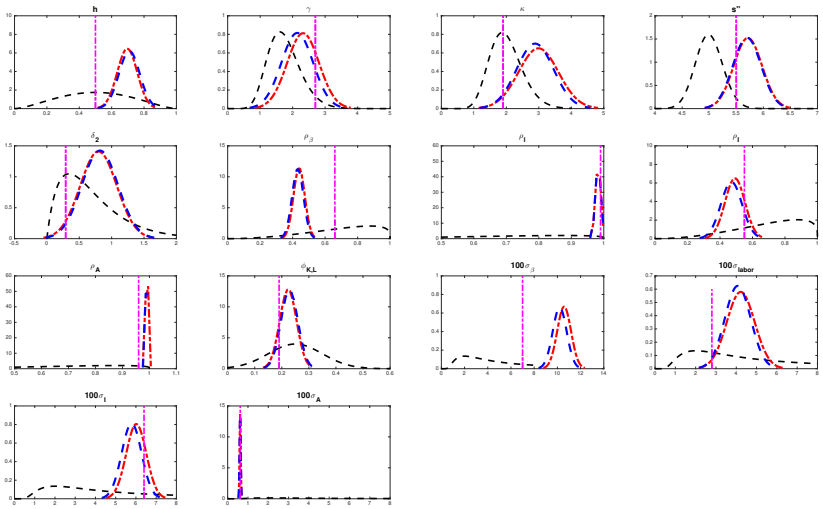

Figure 3: Non-policy parameters: prior(black) vs. FIRE posterior (red) vs. IIAU posterior (blue) vs. Leeper et al. (2010) posterior means (magenta)

5.1.2 Comparing FIRE and IIAU posteriors

Figure 3 and 4 also plot the prior and posterior probability density functions of non-policy and policy parameters that appeared in both FIRE and IIAU models. Across the two model specifications I get similar posterior distributions for the common parameters $\mathcal{P}$.

Table 6 reports both prior and posterior means and 90% credible intervals for the standard deviations of the perceived "shocks"(the $\sigma_{X,\eta}$'s) in the linearized IIAU model. Figure 5 visualizes those prior and posterior distributions. Posterior draws' masses are concentrated at the left end of the prior distributions. However, the MH samplers also accept some large draws of $\sigma_{X,\eta}$'s, resulting positive skewed posteriors with fat tails.

Table 7 reports the log marginal data densities for both FIRE and IIAU models. Geweke (1999) modified harmonic mean estimator are used to calculate log marginal data densities.[25] Bayes factors relative to the linearized

[25]The truncation parameter for the harmonic mean estimator is set to 0.5.

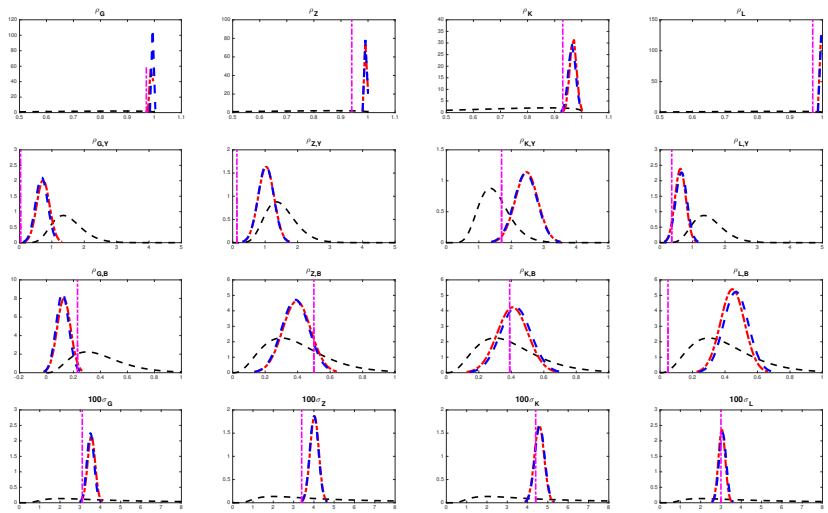

Figure 4: Policy parameters: prior(black) vs. FIRE posterior (red) vs. IIAU posterior (blue) vs. Leeper et al. (2010) posterior means (magenta)

IIAU model are also reported in Table 7. The linearized IIAU model has a slightly better model fit than the FIRE model but data do not strongly prefer one model specification over another. A linearized IIAU model with restrictions $\sigma_{\eta,G} = \sigma_{\eta,Z} = \sigma_{\eta,K} = \sigma_{\eta,L}$ is also estimated and its log-marginal data density is reported in Table 7. Bayes factors indicate the three model specifications are close to observational equivalent.

5.2 Simulation

This section simulates the consequences of low-frequency uncertainties by revisiting the nonlinear IIAU models solved in Section 4.3 and drawing parameters from posteriors. To downplay the role of initial beliefs I assume households' confidence matrix $P_{t+1|t}$ has already converges to P according to (33). Households' beliefs are now completely characterized by $\xi_{t+1|t}$, Appendix B.6 shows the endogenous economic variables Y are influenced by $\xi_{t+1|t}$ in the following way

$$Y_{t+1} = H(\mathcal{H}(\xi_{t+1|t}))Y_t + M(\mathcal{H}(\xi_{t+1|t}))\epsilon_{t+1} \tag{34}$$

Parameter	**Priors**		**Posteriors**	
	mean	90% C.S.	mean	90% C.S.
Std. dev. of $\eta_{X,t}$				
$100\sigma_{\eta,G}$, $\eta_{G,t}$ std.	0.4	[0.14, 0.78]	0.58	[0.19, 1.44]
$100\sigma_{\eta,Z}$, $\eta_{G,t}$ std.	0.4	[0.14, 0.78]	0.98	[0.18, 3.64]
$100\sigma_{\eta,K}$, $\eta_{G,t}$ std.	0.4	[0.14, 0.78]	0.34	[0.12, 0.80]
$100\sigma_{\eta,L}$, $\eta_{G,t}$ std.	0.4	[0.14, 0.78]	0.59	[0.15, 2.10]

Table 6: Prior and posterior distributions of perceived "shocks" in the IIAU model: means and 90-percent credible sets.

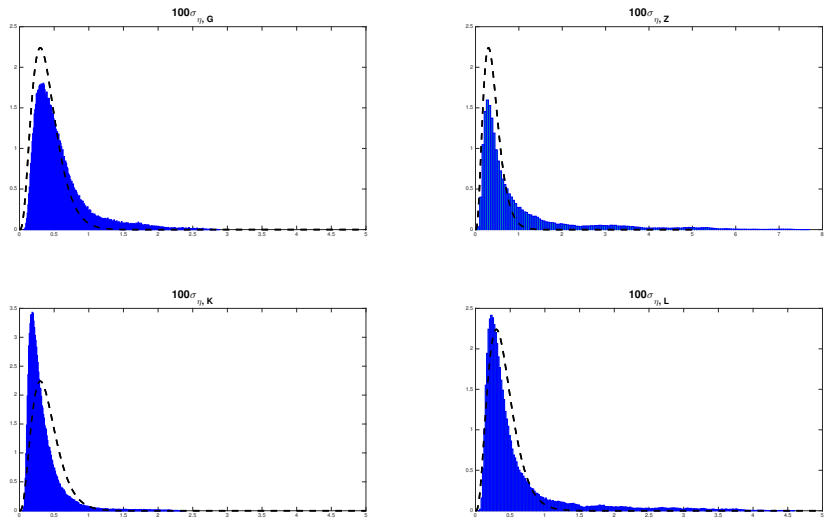

Figure 5: Prior(dashed black) vs. posterior(solid blue) distributions of std. of perceived "shocks".

Model specification	Log-marginal data density	Bayes factor
FIRE	-410.39	exp(0.83)
Linearized IIAU	-409.56	1.0
Linearized IIAU, restrict $\sigma_{\eta,G} = \sigma_{\eta,Z} = \sigma_{\eta,K} = \sigma_{\eta,L}$	-409.82	exp(0.26)

Table 7: Model fit comparisons

where $\mathcal{H}$ provides a nonlinear mapping between households' beliefs $\xi_{t+1|t}$ and their perceived steady states of the economy. H, M are two nonlinear functions which take households' perceived steady states as input and characterize policy functions of the state of the economy.

I first characterize the impacts of low-frequency uncertainties by plotting impulse response functions(IRFs) of economic variables with respect to various fiscal shocks. (34) indicates these IRFs are state-dependent and depends directly on households' current beliefs $\xi_{t+1|t}$. Effects of fiscal policy are complicated since not only will they interact and influence endogenous economic variables, realized fiscal instruments(along with output) will also shape households' beliefs on fiscal targets, which will further amplify or diminish the effects of fiscal policy.

5.2.1 IRFs when initial beliefs match actual fiscal targets

Figure 6 to 9 plot impulse responses following a temporary one standard deviation exogenous increase in each fiscal instrument under both FIRE and IIAU models when households' initial beliefs $\xi_{1|0}$ on fiscal targets are correct. The impulse responses of households' perceived steady states of the economy in the IIAU model are also plotted. The single shock hits the economy at $t = 1$. The dashed lines are generated with the $5th$ and $95th$ percentiles draws based on the posterior distributions. The solids line are generated with the mean estimates of the posterior distributions.

These graphs give insight on how households inside the economy perceive movements of fiscal variables with/without low-frequency fiscal uncertainty when effects of initial beliefs are isolated. High persistence of fiscal variables

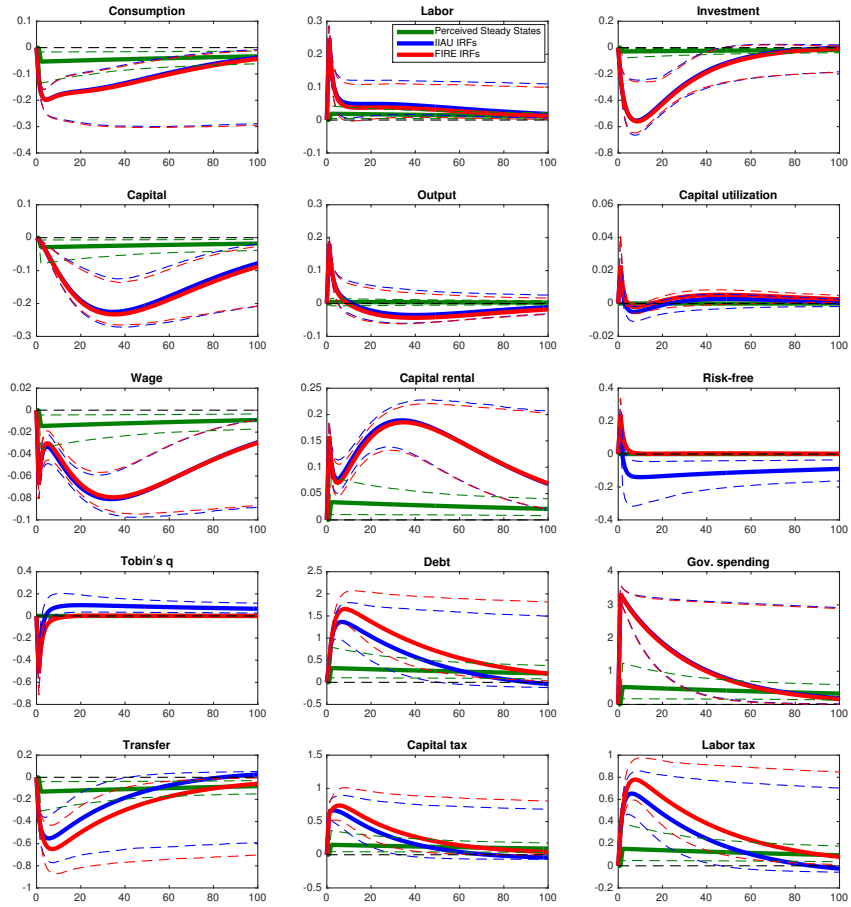

Figure 6: Estimated impulse responses to a one standard deviation increase in government spending shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid line is the mean impulse response; the dashed lines are the 5% and 95% posterior intervals. The x-axis measures quarters and y-axis measures percentages. Households' initial beliefs satisfy $\Gamma_{X,t} = \Gamma_X^*$ for $X = \{G, Z, K, L\}$.

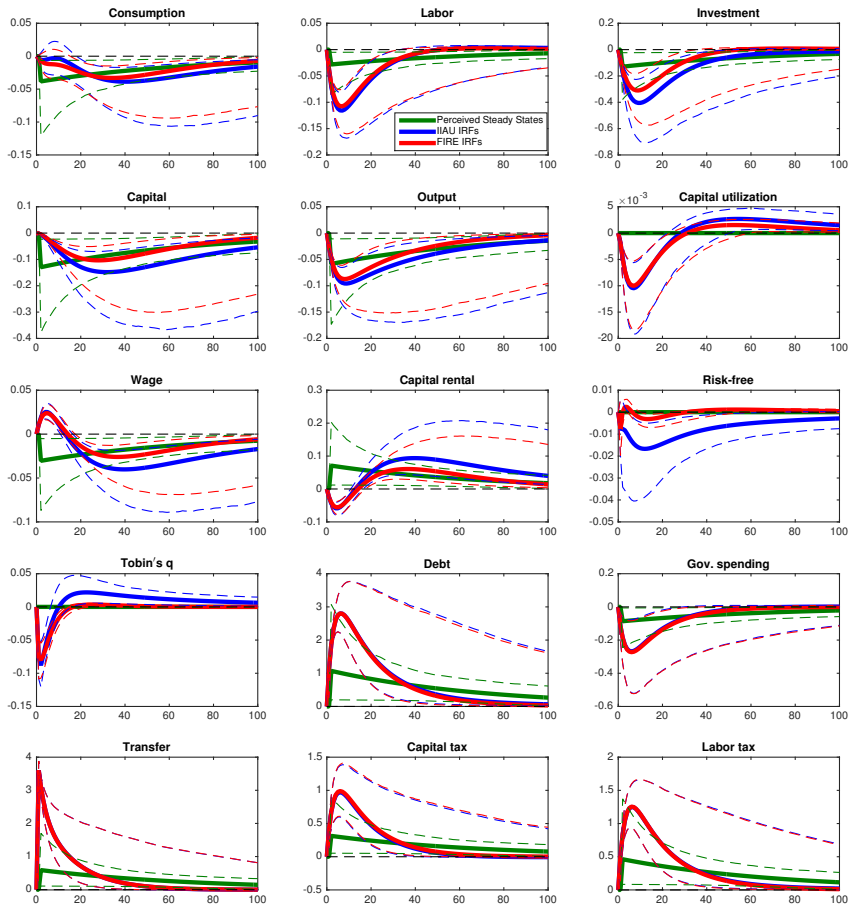

Figure 7: Estimated impulse responses to a one standard deviation increase in lump-sum transfers shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid line is the mean impulse response; the dashed lines are the 5% and 95% posterior intervals. The x-axis measures quarters and y-axis measures percentages. Households' initial beliefs satisfy $\Gamma_{X,t} = \Gamma_X^*$ for $X = \{G, Z, K, L\}$.

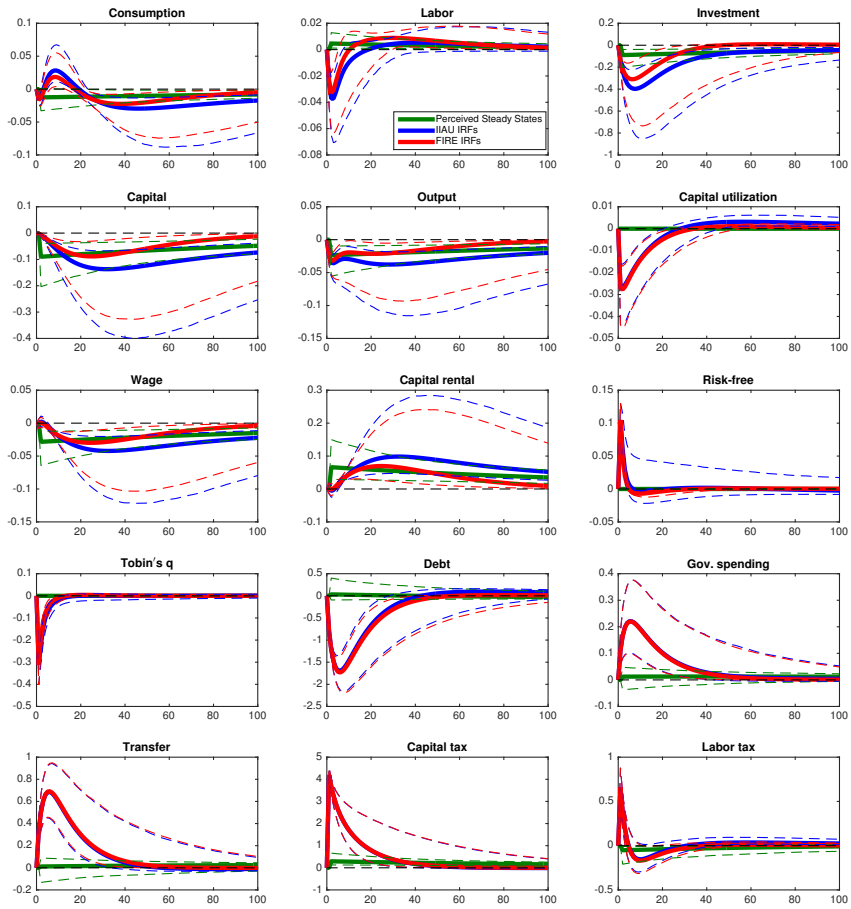

Figure 8: Estimated impulse responses to a one standard deviation increase in capital tax shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid line is the mean impulse response; the dashed lines are the 5% and 95% posterior intervals. The x-axis measures quarters and y-axis measures percentages. Households' initial beliefs satisfy $\Gamma_{X,t} = \Gamma_X^*$ for $X = \{G, Z, K, L\}$.

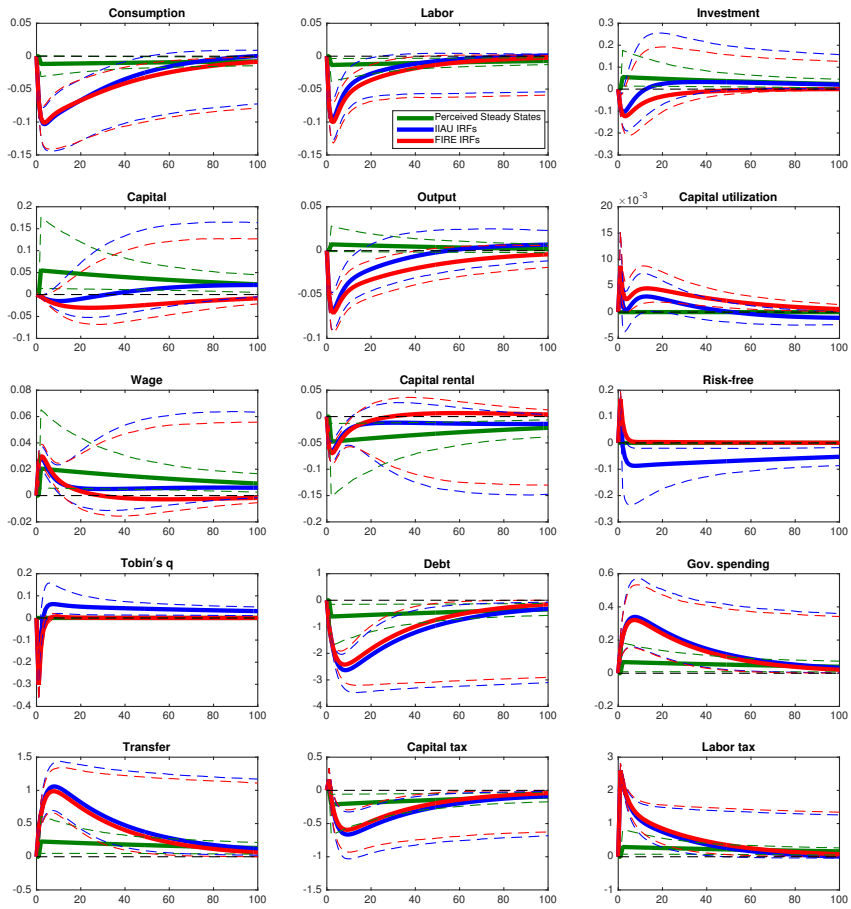

Figure 9: Estimated impulse responses to a one standard deviation increase in labor tax shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid line is the mean impulse response; the dashed lines are the 5% and 95% posterior intervals. The x-axis measures quarters and y-axis measures percentages. Households' initial beliefs satisfy $\Gamma_{X,t} = \Gamma_X^*$ for $X = \{G, Z, K, L\}$.

plays a crucial role. Except for ρ_K in capital tax rule, which has slightly smaller mean estimates(0.97 in FIRE and 0.96 in IIAU), autoregressive coefficients in government spending, transfers, and labor tax rates rules are all concentrated around 0.995 with tiny variances. In consequence, in Figure 6, 7 and 9 under both the FIRE and the IIAU models many variables do not converge to the deterministic steady states of the economy, even after 25 years. In the IIAU models households are also learning fiscal targets, the small jumps of perceived steady states at $t = 1$ in most impulse responses functions indicate households adjust their beliefs after the shock hitting the economy. These jumps are due to fluctuations of households' perceived fiscal targets. Rather persistent fiscal movements force them to keep their $t = 1$ beliefs for a long time given there are no subsequent shocks. Beliefs slowly return to the actual, deterministic steady states.

These perceived fiscal targets are interconnected by the steady-state IEC conditions. Fortunately, the plotted impulse responses indicate data can shed light on the unobservable households' belief formation. Figure 6 and 7 illustrate the responses of endogenous economic variables following a one standard deviation shock to government spending and transfers. Interestingly, these two figures indicate after a temporary shock in government spending(transfers), households raises their perceived steady-state government spending(transfers) level, but decreases their perceived steady state on transfers(government spending). This pattern is consistent with the observed historical G/Y and Z/Y ratios plotted in Figure 1, where we see seemingly downward trend of government spending to GDP ratios and upward trend of transfers to GDP ratios over the last 60 years. Perceived steady-state capital and labor taxes both increase in Figure 6 and 7, although in different magnitudes, resulting different levels of changes in perceived steady-state debt levels.

On the other hand, Figure 8 and 9 suggest when there is a temporary positive shock to one tax rate(say, capital), then households increase perceived steady state of that tax(capital), but decrease perceived steady state of the other tax instrument(labor). While Figure 9 indicates households' perceived government spending and lump-sum transfers targets jump up after the temporary shock to labor tax, Figure 8 implies both targets could rise or fall after one temporary shock to capital tax. In the end, the perceived debt target increase slightly in Figure 8 but decreases in Figure 9.

In general households' perceived fiscal targets move in the same direction with the corresponding fiscal instruments when initial beliefs matches the actual fiscal targets. There are a few exceptions in Figure 8. After a positive shock to capital tax rate, both capital and labor tax rates increases, due to the positively identified correlation term $\phi_{K,L}$ in (21), (22). However, households' perceived steady-state labor tax rate decreases, indicating they interpret observed spike in labor tax rates as results of positive correlations between capital tax rate and the increment is not sufficient large enough for them to shift their beliefs on steady-state labor tax rate. Perceived debt target increases in Figure 8, although the realized debt series are below its actual steady-state level. The higher perceived debt target in Figure 8 is due to stronger fiscal backing and mostly it comes from higher steady-state capital tax revenue and possibly lower steady-state government spending and lump-sum transfers.

Turning to non-policy real economic variables, Figure 6 suggests there's no much difference between the FIRE and IIAU models to a government spending shock. Figure 7, 8 and 9 however reveal differences manifest over time. In particular, low-frequency fiscal uncertainty in general amplify effects appeared in the FIRE model. Perceived steady-state capital tax rate plays a critical role. For instance, a one standard deviation increase in lump-sum transfers (Figure 7) increases capital tax rate, which depresses output, leading to negative transfers multipliers. With low-frequency fiscal uncertainty, households' perceived steady-state capital tax rate increase, which further depresses capital level, causing larger contractionary effects on output. The detrimental effects of higher capital tax rates on output can be seen more clearly in Figure 8: While all other mean perceived fiscal targets estimates are close to their actual targets(government spending: 0.005%, transfers: -0.013%, labor tax: -0.083%, debt: 0.01%), a 0.315% larger than actual capital tax target leads to persistent lower capital and output levels compared to the FIRE model.

Asset prices(wage rate, capital rental rate, risk-free rate and Tobin's q) display larger differences between the FIRE and IIAU models. The risk-free rate(so does Tobin's q) is an extreme case: In sharp contrast to FIRE models in which the impulse responses of risk-free rate can only have very narrow 90% posterior intervals, the corresponding impulse responses in the IIAU model

have significantly wider 90% posterior intervals. With low-frequency fiscal uncertainty the impacts of fiscal shocks on risk-free rate are far-reaching and long-lived: In Figure 6 and 9 the mean impulse responses of risk-free rates implies they behave as if they are permanently lower. Simulation results not reported in the paper shows risk-free rate can display both upward or downward trends over a long period of time. These differences transmit to other asset prices due to non-arbitrage conditions. Wage and capital rental rates behave likewise in both FIRE and IIAU models. The low-frequency fiscal uncertainty, however, amplify how these prices respond to the fiscal shocks by shifting their own perceived steady states.

Finally, extending the simulation horizon of the IRFs indicate impulse responses of the FIRE model converge to zero while in the IIAU model impulse responses converge to their perceived steady states first. Perceived steady states slowly converge to their actual steady states.

5.2.2 IRFs when initial beliefs deviate from actual fiscal targets

Figure 10 to 13 plot impulse responses when households' initial beliefs at $t = 0$ are not aligned with the actual fiscal targets. The economy sits in its actual steady state at $t = 0$. For illustration I consider two cases where households' initial beliefs $\Gamma_{X,0}$ equals to $0.99\Gamma_X^*$ or $1.01\Gamma_X^*$. The green lines track the evolution of households' beliefs of the economy's steady states while the blue lines track the variables' impulse responses to temporary fiscal shocks. The red lines plot the corresponding impulse responses in the FIRE model.

In sharp contrast to impulse responses plotted in Figure 6 to 9, small deviations of households' initial beliefs from actual fiscal targets yield completely different impulse responses, both qualitatively and quantitatively. There are differences both in the short run and in the long run. Depending on initial beliefs on fiscal targets, the same temporary fiscal shock can either be expansionary or contractionary. A common pattern emerges: Instead of converging to zero directly as in the FIRE case, impulse responses of variables in the IIAU model converge to their perceived steady states first. The perceived steady states gradually go back to the actual steady states provided there are no subsequent shocks.

For example, in Figure 10, the solid greens lines indicate households'

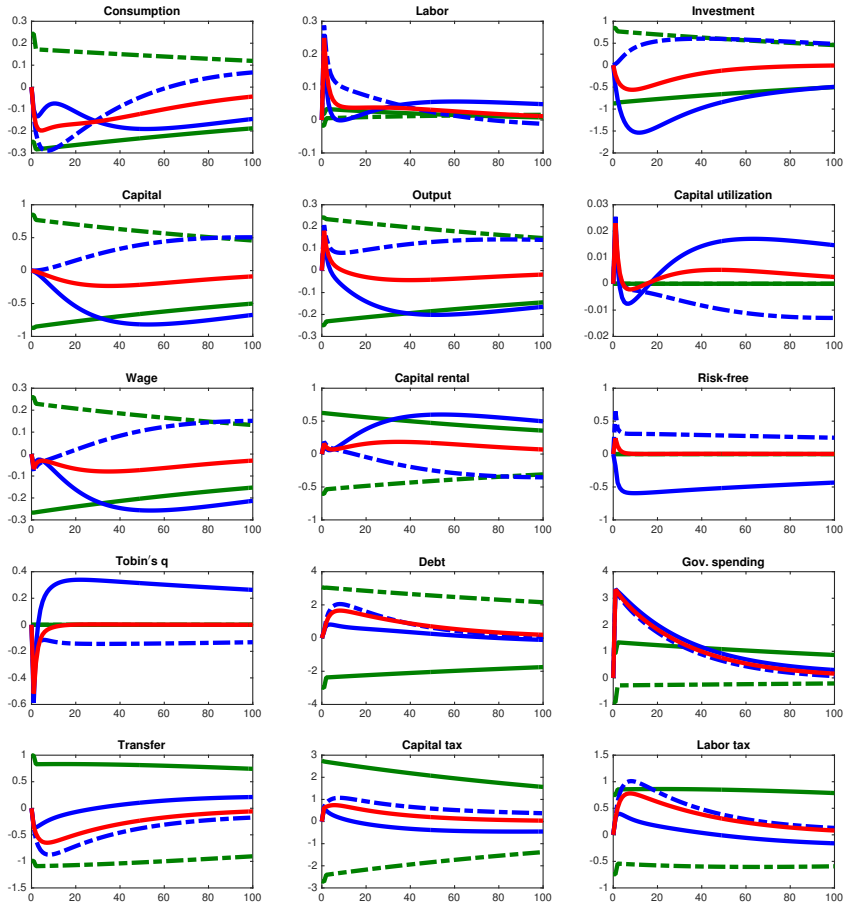

Figure 10: Estimated mean impulse responses to a one standard deviation increase in government spending shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 0.99\Gamma_X^*$; the dashed lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 1.01\Gamma_X^*$ for $X = \{G, Z, K, L\}$. The x-axis measures quarters and y-axis measures percentages.

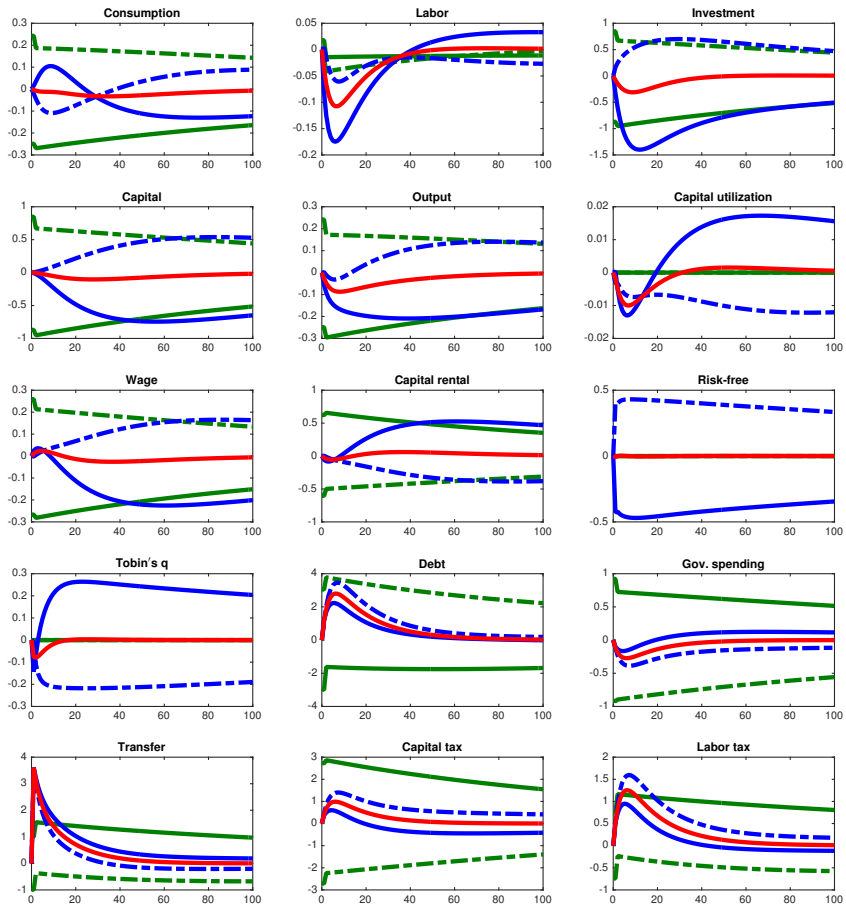

Figure 11: Estimated mean impulse responses to a one standard deviation increase in transfers shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 0.99\Gamma_X^*$; the dashed lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 1.01\Gamma_X^*$ for $X = \{G, Z, K, L\}$. The x-axis measures quarters and y-axis measures percentages.

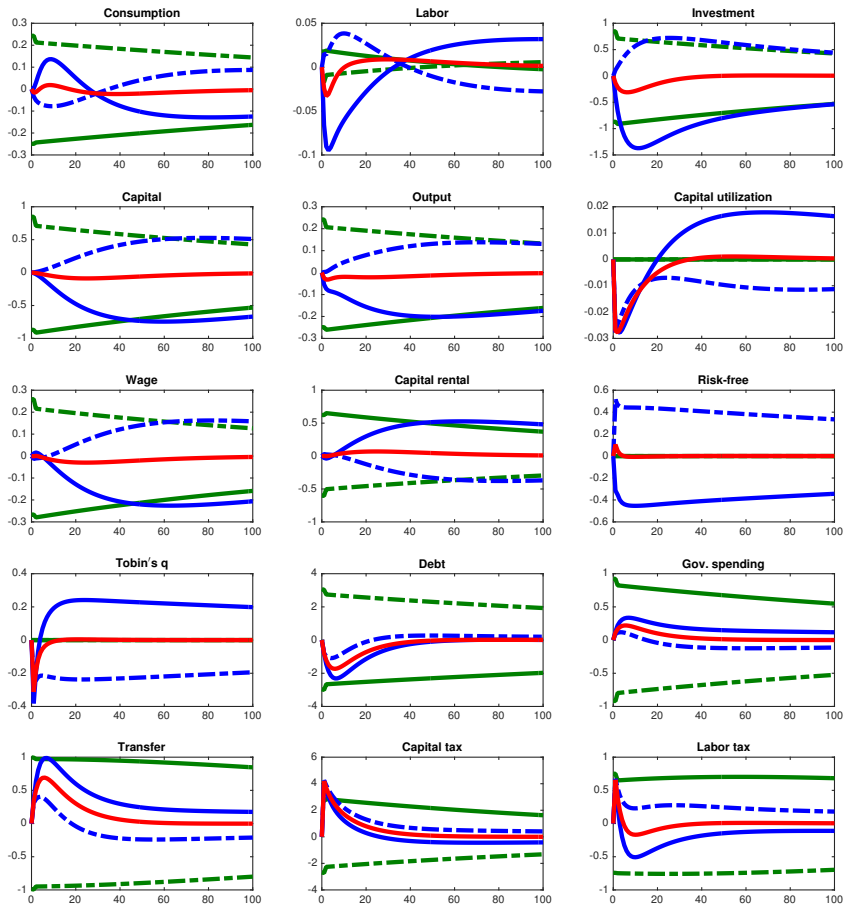

Figure 12: Estimated mean impulse responses to a one standard deviation increase in capital tax shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 0.99\Gamma_X^*$; the dashed lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 1.01\Gamma_X^*$ for $X = \{G, Z, K, L\}$. The x-axis measures quarters and y-axis measures percentages.

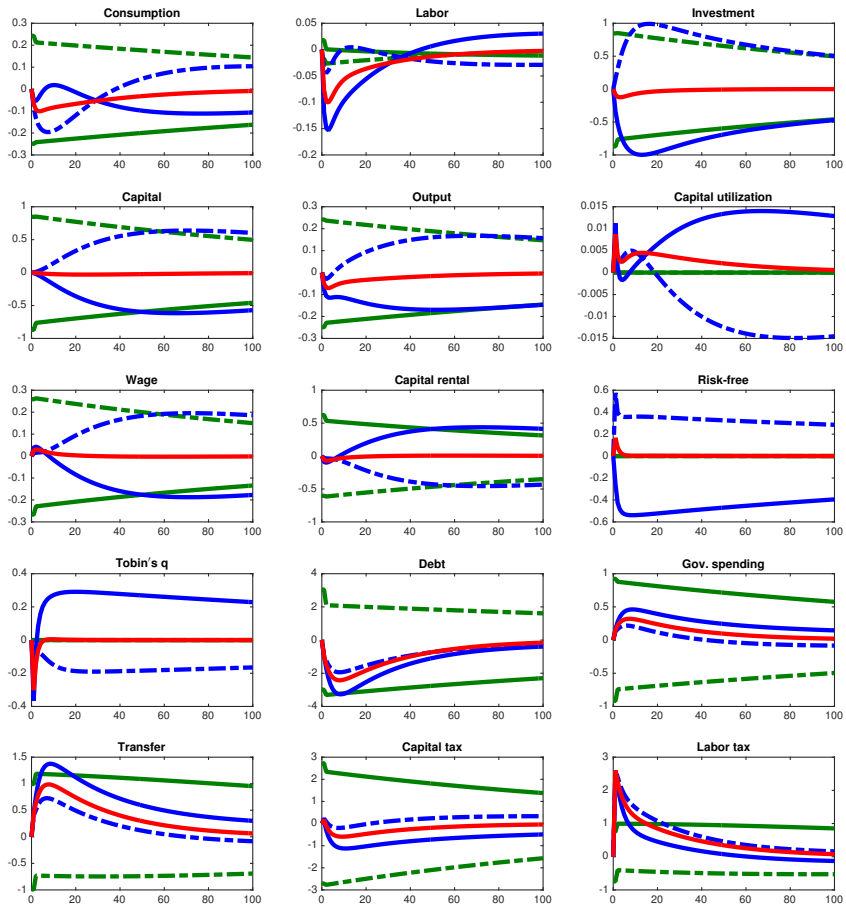

Figure 13: Estimated mean impulse responses to a one standard deviation increase in labor tax shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The solid lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 0.99\Gamma_X^*$; the dashed lines are when households' initial beliefs satisfy $\Gamma_{X,0} = 1.01\Gamma_X^*$ for $X = \{G, Z, K, L\}$. The x-axis measures quarters and y-axis measures percentages.

perceived steady-state government spending, transfers, capital and labor tax rates right before the shock hitting the economy are $0.96\%, 1.00\%, 2.75\%$ and 0.75% higher than the actual fiscal targets. In consequence, the implied perceived steady-state debt target is 2.96% lower than its actual target. Such combination of fiscal targets implies a lower steady-state capital and output level. It follows although the temporary government spending shock is still expansionary in the short run, in the long run it becomes more contractionary compared to the FIRE case. On the other hand, the dashed green lines indicate households' initial perceived steady-state government spending, transfers, capital and labor tax rates are 0.91%, 0.99%, 2.69% and 0.74% lower than the actual fiscal targets. The steady-state IEC implies a perceived steady-state debt target 3.07% higher than its actual target. Households now perceive a higher steady-state output level. The same government spending shock becomes expansionary both in the short run and in the long run.

It is important to emphasize the large discrepancies between the FIRE and the IIAU impulse responses are originated from different beliefs of fiscal targets at $t = 0$ before the shock hitting the economy at $t = 1$. Different perceptions of fiscal targets, combined with the non-policy model structure $\mathcal{M}^{NP}$, imply different steady states of the economy. In consequence, to the policy maker/econometrician/economists who have perfect information on fiscal targets, they would believe all economic variables is on its steady states. On the other hand, to the households inside the economy who have a different views of fiscal targets, their perceived steady states of the economy are different. Different perceptions of steady states drive the large differences in the plotted impulse responses.

The above impulse responses imply to evaluate the effects of fiscal policy, tracking households' beliefs on fiscal targets is of first order importance. Simply ignoring low-frequency fiscal uncertainty and considering the FIRE scenario may lead to qualitatively and quantitatively misleading policy evaluations across all horizons.

In the following section I apply a Rao-Blackwellized particle filter(RBPF) developed by Schon et al. (2005) to extract households' beliefs on fiscal targets by using actual U.S. data. In Section 5.1 the same dataset has been used to estimate the structural parameters. The RBPF is an application of sequential Monte Carlo(SMC) methods which are a class of simulation-based

algorithms to solve optimal filtering problems for non-linear non-Gaussian state space models. For a short introduction on SMC methods, see Appendix D.1. Appendix D.2 introduces the Schon et al. (2005) RBPF and illustrates how it could be applied to the nonlinear IIAU model. The key to apply this RBPF is to recognize the nonlinear IIAU model solution is conditionally linear on households' beliefs $[\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}]$, which contain sufficient information to pin down households' perceived fiscal targets.

5.3 Filtering

Figure 14 and 15 plot various observed fiscal variable to GDP ratios from 1959Q3 to 2016Q1, actual fiscal targets which are calibrated by the econometricians along with filtered households' perceived fiscal targets for U.S. data from 1980Q1 to 2016Q1 when parameters are fixed at the posterior means reported in Section 5.1. Filtered beliefs indicate overtime there are 10% to 20% of deviations of households' perceived fiscal targets from calibrated fiscal targets. For example, households' perceived steady-state government spending and transfers to GDP ratios are consistently lower than the calibrated values starting from the 1980s. Average capital and labor tax rates are not directly observable and their perceived steady-states are derived from observed capital and labor tax revenues. While the perceived steady-state capital tax rates display fluctuations around the calibrated value, perceived labor tax rate targest are moving at a level around 10% lower than its calibrated value.

Figure 15 plot observed debt-to-GDP ratios along with filtered households' beliefs. Compared to other fiscal targets, perceived steady-state debt-to-GDP ratios display relatively larger deviations. For instance, during mid 1990s, households' perceived debt target is close to 41% of GDP, morn than 20% of the calibrated steady-state debt-to-GDP ratio 33.96%. Recent rapid increase in debt-to-GDP ratios due to large expansionary fiscal policies (e.g., the American Recovery and Reinvestment Act of 2009 and the extended Bush tax cut from 2011) cause households to increase their perceived steady-state debt-to-GDP ratio from 36% to 48% from 2009 to 2011, an increment of more than 30%. At the end of 2011 households' perceived debt target is 41% higher than the calibrated value, which is a record high. The perceived debt-to-GDP ratios then subside from 2011 to 2016, partly because higher perceived steady-state capital tax revenues and lower perceived steady-state

government spending. These perceived steady-state fiscal targets, combined with the steady-state IEC, imply the government can support a higher debt-to-GDP ratios in the long run. Notice that during this period the perceived steady-state labor tax revenues actually decrease around 10% while the perceived transfers target remain relatively constant. If the perceived labor tax revenue targets were not decreasing, all else equal, the steady-state IEC would imply a higher perceived debt-to-GDP ratio targets.

Figure 16 illustrates the impacts of low-frequency fiscal uncertainty by simulating both the FIRE and the IIAU economies after a negative four standard deviation decrease in technology shock. All parameter values are fixed at the means of posterior draws. The economy sits in its actual deterministic steady state at $t = 0$. The single technology shock hits the economy at $t = 1$. Due to the high persistence of technology shock ($\rho_A = 0.99$), such a large negative shock generates far-reaching contractionary effects on the real economy. In the FIRE model output remains 1.5% lower than its steady-state level, even after 25 years. To build a connection with the recent Great Recession starting from 2007Q4, the RBPF filtered beliefs at 2007Q3 is used as the households' initial beliefs at $t = 0$ in the IIAU model. The large technology shock thus mimics the Great Recession the economy has suffered.

The RBPF extracted 2007Q3 beliefs imply households' perceived steady-state debt, government spending, transfers, capital and labor taxes deviate $13.6\%, -11.2\%, -2.8\%, 3.2\%, -9.8\%$ from their actual targets. The implied steady-state values of non-fiscal economic variables display much smaller deviations. As Figure 16 shows, at $t = 0$ households' perceived steady-state output is 0.35% lower than its actual steady state, perceived steady-state labor hours is -0.05% smaller. Due to higher perceived capital tax target, perceived steady-state investment and capital levels are around 1.06% lower than its FIRE counterparts. Less saving combined with other perceived steady states implies a perceived steady-state consumption level 1.23% higher than its actual FIRE value.

With all other shocks eliminated, Figure 16 illustrates under the IIAU model the same negative technology shock can generate deeper recessions. For simplicity, all variables at $t = 0$ are fixed at their FIRE deterministic steady state values. Recession decreases capital and labor tax revenues,

causing debt to rise fast under both FIRE and IIAU. Under IIAU the risk-free rate increases much more than its FIRE counterpart. Higher borrowing cost largely increase government's debt level. Instead of conducting expansionary fiscal policies to fight of the recession, higher debt levels force the fiscal authority to decrease government spending and transfer levels and to increase both taxes. Higher capital and labor taxes further depresses labor supply and investment activities, which lead to a deeper recession.

6 Conclusion

This paper defines and explores consequences of low-frequency fiscal uncertainty in an RBC model with a detailed fiscal sector in which households are learning unknown fiscal targets. Highly persistent fiscal movements cause households to suspect fiscal targets may be time-varying, even though the underlying fiscal targets are all time-invariant. Time-varying perceived fiscal targets bring far-reaching impacts into the economy. While the low-frequency fiscal uncertainty don't play significant roles when households' perceived fiscal targets match the actual fiscal targets, small deviations of household' beliefs largely change effects of temporary fiscal shocks across all horizons. These results highlight the importance of households' beliefs on fiscal targets and how households' beliefs evolve is crucial in evaluating current fiscal policies.

Utilizing a Rao-Blackwellized particle filter, data suggest there are large deviations of households' beliefs from calibrated fiscal targets. Using the extracted beliefs for 2007Q3, low-frequency fiscal uncertainty may have deepened the Great Recession. These results call for more transparent communications on fiscal targets.

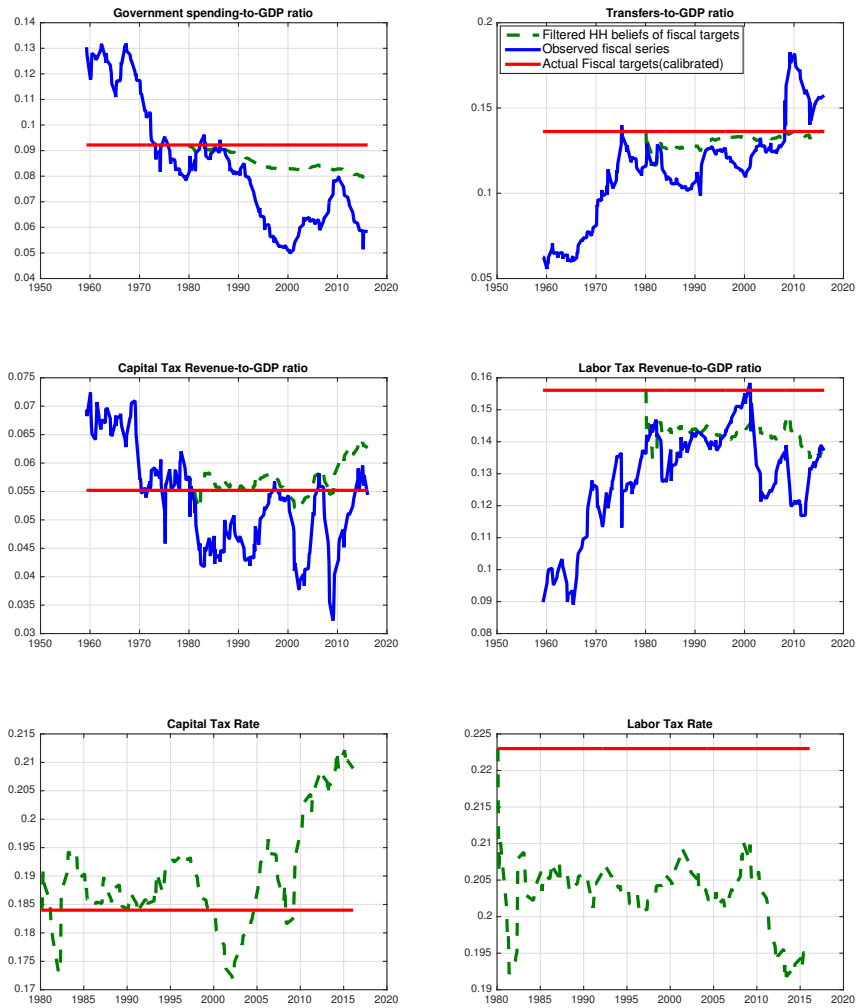

Figure 14: Observed U.S. fiscal series from 1959Q3 to 2016Q1(solid blue) and filtered households' perceived fiscal targets from 1980Q1 to 2016Q1(dashed green), along with the calibrated fiscal targets (solid red). Parameters are fixed at the posterior means.

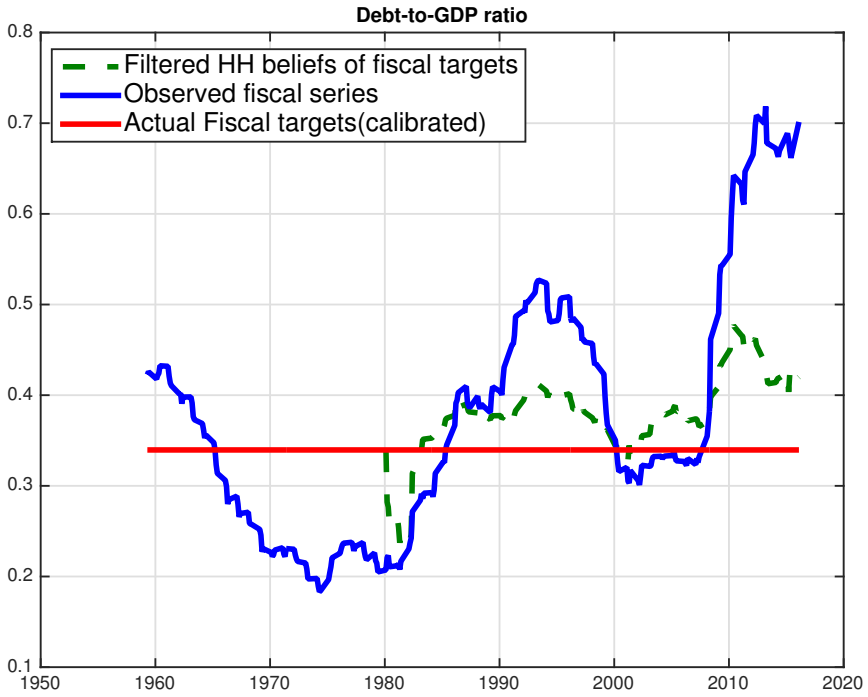

Figure 15: Observed U.S. debt-to-GDP ratios from 1980Q1 to 2016Q1(solid blue) and filtered households' perceived steady-state debt-to-GDP ratio from 1980Q1 to 2016Q1(dashed green), along with the calibrated debt-to-GDP ratio(solid red). Parameters are fixed at the posterior means.

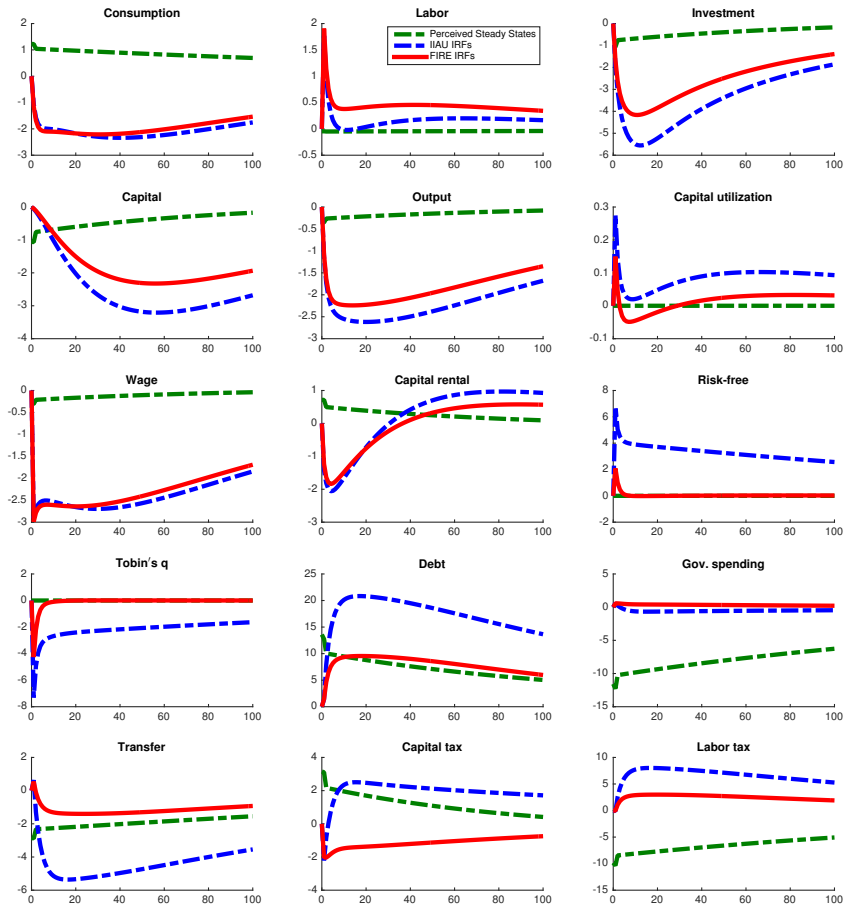

Figure 16: Impulse responses to a four standard deviation decrease in technology shock in FIRE(red) and IIAU(blue) models, along with impulse responses of households' perceived steady states of the economy in the IIAU model(green). The single shock hits the economy at $t = 1$. RBPF filtered beliefs at 2007Q3 is used as households' initial beliefs at $t = 0$. The x-axis measures quarters and y-axis measures percentages.

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A Appendix A.

A.1 Solving the second-order difference equation (6) or (7)

I first consider the FIRE case and solve (6). Given AR(1) technology (1) and $i.i.d$ tax rule (2), $\mathbb{E}_t\hat{A}_{t+1} = \rho\hat{A}_t$ and $\mathbb{E}_t\hat{\tau}_{t+1} = 0$. Define $A = \alpha\beta\gamma(1-\tau^*)$, $B = [(\alpha-1)(1-\alpha\beta(1-\tau^*)) - \alpha\gamma - \alpha\beta\gamma(1-\tau^*)]$, $C = \alpha\gamma$ and $D = \gamma\rho - [1-\alpha\beta(1-\tau^*)]\rho - \sigma$, it is easy to see $\hat{K}_t$ solves the following difference equation

$$A\mathbb{E}_t\hat{K}_{t+1} + B\hat{K}_t + C\hat{K}_{t-1} = D\hat{A}_t$$

Guessing $\hat{K}_t = F(L)\epsilon_t^A$ where L is the lag operator and utilizing the Wiener-Kolmogrov formula give

$$\mathbb{E}_t\hat{K}_{t+1} = L^{-1}[F(L) - F_0]\epsilon_t^A$$

Along with $\hat{A}_t = \frac{1}{1-\rho L}\epsilon_t^A$ and $\hat{K}_{t-1} = LF(L)\epsilon_t^A$, the difference equation implies in frequency domain

$$Az^{-1}[F(z) - F_0] + BF(z) + CzF(z) = \frac{D}{1-\rho z}$$

where F_0 is a free parameter to be pinned down to guarantee the stationarity of $\hat{K}_t$. Rearranging terms gives

$$(z^2 + \frac{B}{C}z + \frac{A}{C})F(z) = \frac{D}{C}\frac{z}{1-\rho z} + \frac{A}{C}F_0$$

Existence requires roots to the quadratic equation $z^2 + \frac{B}{C}z + \frac{A}{C} = 0$ are real. Let the two roots be denoted z_1, z_2. Factoring $z^2 + \frac{B}{C}z + \frac{A}{C}$ as $(z-z_1)(z-z_2)$ gives

$$F(z) = \frac{\frac{D}{C}\frac{z}{1-\rho z} + \frac{A}{C}F_0}{(z-z_1)(z-z_2)}$$

Uniqueness requires one root, say z_1's module is less than 1 while the other root, z_2's module is strictly great than 1, this imposes a restriction on F_0, in particular,

$$\frac{D}{C}\frac{z_1}{1-\rho z_1} + \frac{A}{C}F_0 = 0$$

In consequence,

$$F(z) = \frac{\frac{D}{C}\frac{z}{1-\rho z} - \frac{D}{C}\frac{z_1}{1-\rho z_1}}{(z-z_1)(z-z_2)} = \frac{D}{C(1-\rho z_1)}\frac{1}{(z-z_2)(1-\rho z)}$$

This implies

$$\hat{K}_t = \frac{1}{z_2}\hat{K}_{t-1} - \frac{D}{C(1-\rho z_1)z_2}\hat{A}_t$$

Define $f_1(\tau^*) = \frac{1}{z_2}, f_2(\tau^*) = -\frac{D}{C(1-\rho z_1)z_2}$ give

$$\hat{K}_t = f_1(\tau^*)\hat{K}_{t-1} + f_2(\tau^*)\hat{A}_t$$

The anticipated utility approach allows treating $\tau_t^{\circledast}$ as a constant when solving (7). Repeating the same procedure thus will give

$$\widetilde{K}_t^{\circledast} = f_1(\tau_t^{\circledast})\widetilde{K}_{t-1} + f_2(\tau_t^{\circledast})\widetilde{A}_t$$

This completes the derivations.

B Appendix B.

B.1 Equilibrium conditions of the RBC model

There are 8 exogenous innovations

$$\{\epsilon_t^{\beta}, \epsilon_t^{\ell}, \epsilon_t^{I}, \epsilon_t^{A}, \epsilon_t^{G}, \epsilon_t^{Z}, \epsilon_t^{\tau_K}, \epsilon_t^{\tau_L}\}$$

along with 25 endogenous variables[26]

$$\begin{aligned}
\text{Real allocations: } & C_t, L_t, I_t, K_t, Y_t, v_t \\
\text{Real prices: } & W_t, R_t^K, R_t, q_t \\
\text{Fiscal instruments: } & B_t, G_t, Z_t, \tau_t^K, \tau_t^L, T_t^K, T_t^L \\
\text{AR(1) shocks: } & u_t^{\beta}, u_t^{\ell}, u_t^{I}, u_t^{A}, u_t^{G}, u_t^{Z}, u_t^{\tau_K}, u_t^{\tau_L}
\end{aligned}$$

[26] $q_t = \frac{\mu_t}{\lambda_t}$ is the Tobin's Q. T_t^K and T_t^L are capital and labor tax revenues. v_t measures the capital utilization in time t.

in the model. The following 18 equations along with the 8 equations defined in the four fiscal rules (19), (20), (21) and (22) characterize the equilibrium conditions of the RBC model.

$$u_t^{\beta}(C_t - hC_{t-1})^{-\gamma} = E_t^{\mathcal{I}} \beta R_t u_{t+1}^{\beta}(C_{t+1} - hC_t)^{-\gamma}$$
$$u_t^{\ell} L_t^{1+\kappa}(1+\tau_C^*) = (C_t - hC_{t-1})^{-\gamma}(1-\tau_t^L)(1-\alpha)Y_t$$
$$q_t = \beta E_t^{\mathcal{I}} \frac{u_{t+1}^{\beta}(C_{t+1} - hC_t)^{-\gamma}}{u_t^{\beta}(C_t - hC_{t-1})^{-\gamma}} \left\{ (1-\tau_{t+1}^K)\frac{\alpha Y_{t+1}}{K_t} + q_{t+1}[1-\delta(v_{t+1})] \right\}$$
$$\frac{\alpha Y_t(1-\tau_t^K)}{v_t K_{t-1}} = q_t[\delta_1 + \delta_2(v_t - 1)]$$
$$1 = q_t \left\{ [1 - s_t(\cdot)] - s'(\cdot)\frac{u_t^I I_t}{I_{t-1}} \right\} + \beta E_t^{\mathcal{I}} \left\{ \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} s'_{t+1}(\cdot) \left(\frac{u_{t+1}^I I_{t+1}}{I_t} \right)^2 \right\}$$
$$W_t = \frac{(1-\alpha)Y_t}{L_t}$$
$$R_t^K = \frac{\alpha Y_t}{v_t K_{t-1}}$$
$$T_t^K = \tau_t^K \alpha Y_t$$
$$T_t^L = \tau_t^L (1-\alpha) Y_t$$
$$Y_t = u_t^A (v_t K_{t-1})^{\alpha} L_t^{1-\alpha}$$
$$K_t = [1-\delta(v_t)]K_{t-1} + \left[1 - s(\frac{u_t^I I_t}{I_{t-1}}) \right] I_t$$
$$B_t + T_t^K + T_t^L + \tau_C^* C_t = R_{t-1}B_{t-1} + G_t + Z_t$$
$$Y_t = C_t + I_t + G_t$$
$$\log(u_t^{\beta}) = \rho_{\beta} \log(u_{t-1}^{\beta}) + \sigma_{\beta}\epsilon_t^{\beta}$$
$$\log(u_t^{\ell}) = \rho_{\ell} \log(u_{t-1}^{\ell}) + \sigma_{\ell}\epsilon_t^{\ell}$$
$$\log(u_t^{I}) = \rho_{I} \log(u_{t-1}^{I}) + \sigma_{I}\epsilon_t^{I}$$
$$\log(u_t^{A}) = \rho_{A} \log(u_{t-1}^{A}) + \sigma_{A}\epsilon_t^{A}$$

B.2 Observables $\mathcal{O}_t$ and parameters $\mathcal{P}$ of the RBC model

$\mathcal{O}_t$ include the entire history of variables observable to the household before making decisions at time t. Let $x^t = \{x_t, x_{t-1}, x_{t-2}, ...\}$ denote the history of

variable x up to time t, $\mathcal{O}_t$ is given by

$$\begin{aligned}\mathcal{O}_t &= \{(u^\beta)^t, (u^\ell)^t, (u^A)^t, (u^I)^t, W^t, (R^K)^t, R^t, q^t, v^t, C^{t-1}, L^{t-1}, I^{t-1}, K^{t-1}, \\ &\quad Y^{t-1}, B^{t-1}, G^{t-1}, Z^{t-1}, (\tau^K)^{t-1}, (\tau^L)^{t-1}, (TK)^{t-1}, (TL)^{t-1}\}.\end{aligned}$$

$\mathcal{P}$ contains all parameters appeared in the FIRE model. It is a 43×1 vector and is given by

$$\begin{aligned}\mathcal{P} &= \{\beta, \delta_0, \delta_1, \alpha, \tau_C^*, G^*, B^*, \tau_K^*, \tau_L^*, h, \gamma, \kappa, s'', \delta_2, \rho_\beta, \rho_\ell, \rho_I, \rho_A, \\ &\quad \rho_G, \rho_Z, \rho_K, \rho_L, \phi_{K,L}, \phi_G, \phi_Z, \phi_K, \phi_L, \rho_{G,Y}, \rho_{Z,Y}, \rho_{K,Y}, \rho_{L,Y}, \\ &\quad \rho_{G,B}, \rho_{Z,B}, \rho_{K,B}, \rho_{L,B}, \sigma_\beta, \sigma_\ell, \sigma_I, \sigma_A, \sigma_G, \sigma_Z, \sigma_K, \sigma_L\}.\end{aligned}$$

B.3 Actual steady state of the economy

I use X^* to denote variable X's actual, deterministic steady-state values. The following 11 variables share the same steady state under FIRE and IIAU,

$$v^* = 1; q^* = 1; R^* = \frac{1}{\beta};$$
$$u_\beta^* = u_\ell^* = u_I^* = u_A^* = 1; u_G^* = u_Z^* = u_K^* = u_L^* = 0;$$

Steady-state government spending, capital income and labor income tax rates are denoted by G^*, τ_K^* and τ_L^*. The following 9 equations jointly determine $\{C^*, L^*, I^*, Y^*, K^*, R_K^*, W^*, T_K^*, T_L^*\}$,

$$\begin{gathered}(1+\tau_C^*)(L^*)^{1+\kappa} = (C^* - hC^*)^{-\gamma}(1-\tau_L^*)(1-\alpha)Y^* \\ R_K^* = \delta_1/(1-\tau_K^*) \\ I^* = \delta_0 K^* \\ Y^* = C^* + I^* + G^* \\ Y^* = (K^*)^\alpha (L^*)^{1-\alpha} \\ R_K^* = \alpha Y^*/K^* \\ W^* = (1-\alpha)Y^*/L^* \\ T_K^* = \alpha \tau_K^* Y^* \\ T_L^* = (1-\alpha)\tau_L^* Y^*\end{gathered}$$

The steady-state output level Y^* thus only depends on three fiscal targets G^*, τ_K^* and τ_L^*. Along with the steady-state lump-sum transfers Z^*, the steady-state debt level B^* must satisfy

$$B^* + T_K^* + T_L^* + \tau_C^* C^* = R^* B^* + G^* + Z^*.$$

B.4 Perceived steady state of the economy

I use $X_t^{\circledast}$ to denote variable X's perceived steady-state values under IIAU. As argued in (B.3),

$$v_t^{\circledast} = 1; q_t^{\circledast} = 1; R_t^{\circledast} = \frac{1}{\beta};$$
$$u_{\beta,t}^{\circledast} = u_{\ell,t}^{\circledast} = u_{I,t}^{\circledast} = u_{A,t}^{\circledast} = 1; u_{G,t}^{\circledast} = u_{Z,t}^{\circledast} = u_{K,t}^{\circledast} = u_{L,t}^{\circledast} = 0;$$

The following 14 equations establish a mapping between household's perceived steady states

$$\{C_t^{\circledast}, L_t^{\circledast}, I_t^{\circledast}, Y_t^{\circledast}, K_t^{\circledast}, R_{K,t}^{\circledast}, W_t^{\circledast}, T_{K,t}^{\circledast}, T_{L,t}^{\circledast}, G_t^{\circledast}, Z_t^{\circledast}, \tau_{K,t}^{\circledast}, \tau_{L,t}^{\circledast}, B_t^{\circledast}\}$$

and $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}$.

$$\Gamma_{G,t} = \log(G_t^{\circledast}) + \rho_{G,Y}\log(Y_t^{\circledast}) + \rho_{G,B}\log(B_t^{\circledast})$$
$$\Gamma_{Z,t} = \log(Z_t^{\circledast}) + \rho_{Z,Y}\log(Y_t^{\circledast}) + \rho_{Z,B}\log(B_t^{\circledast})$$
$$\Gamma_{K,t} = \log(\tau_{K,t}^{\circledast}) - \rho_{K,Y}\log(Y_t^{\circledast}) - \rho_{K,B}\log(B_t^{\circledast})$$
$$\Gamma_{L,t} = \log(\tau_{L,t}^{\circledast}) - \rho_{L,Y}\log(Y_t^{\circledast}) - \rho_{L,B}\log(B_t^{\circledast})$$
$$(1+\tau_C^*)(L_t^{\circledast})^{1+\kappa} = (C_t^{\circledast} - hC_t^{\circledast})^{-\gamma}(1-\tau_{L,t}^{\circledast})(1-\alpha)Y_t^{\circledast}$$
$$R_{K,t}^{\circledast} = \delta_1/(1-\tau_{K,t}^{\circledast})$$
$$I_t^{\circledast} = \delta_0 K_t^{\circledast}$$
$$Y_t^{\circledast} = C_t^{\circledast} + I_t^{\circledast} + G_t^{\circledast}$$
$$Y_t^{\circledast} = (K_t^{\circledast})^{\alpha}(L_t^{\circledast})^{1-\alpha}$$
$$R_{K,t}^{\circledast} = \alpha Y_t^{\circledast}/K_t^{\circledast}$$
$$W_t^{\circledast} = (1-\alpha)Y_t^{\circledast}/L_t^{\circledast}$$
$$T_{K,t}^{\circledast} = \alpha\tau_{K,t}^{\circledast}Y_t^{\circledast}$$
$$T_{L,t}^{\circledast} = (1-\alpha)\tau_{L,t}^{\circledast}Y_t^{\circledast}$$
$$B_t^{\circledast} + T_{K,t}^{\circledast} + T_{L,t}^{*} + \tau_C^* C_t^{\circledast} = R_t^{\circledast}B_t^{\circledast} + G_t^{\circledast} + Z_t^{\circledast}.$$

B.5 Establish household's learning state-space model

The state transition equation of the household's learning problem is given by

$$\underbrace{\begin{bmatrix}\Gamma_{G,t+1}\\ \Gamma_{Z,t+1}\\ \Gamma_{K,t+1}\\ \Gamma_{L,t+1}\\ u_{G,t+1}\\ u_{Z,t+1}\\ u_{K,t+1}\\ u_{L,t+1}\end{bmatrix}}_{\xi_{t+1}} = \underbrace{\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&\rho_G&0&0&0\\0&0&0&0&0&\rho_Z&0&0\\0&0&0&0&0&0&\rho_K&0\\0&0&0&0&0&0&0&\rho_L\end{bmatrix}}_{F}\underbrace{\begin{bmatrix}\Gamma_{G,t}\\ \Gamma_{Z,t}\\ \Gamma_{K,t}\\ \Gamma_{L,t}\\ u_{G,t}\\ u_{Z,t}\\ u_{K,t}\\ u_{L,t}\end{bmatrix}}_{\xi_t} + \underbrace{\begin{bmatrix}\eta_{G,t+1}\\ \eta_{Z,t+1}\\ \eta_{K,t+1}\\ \eta_{L,t+1}\\ \sigma_G\epsilon_{G,t+1}\\ \sigma_Z\epsilon_{Z,t+1}\\ \sigma_K\epsilon_{K,t+1}\\ \sigma_L\epsilon_{L,t+1}\end{bmatrix}}_{v_{t+1}} \tag{35}$$

Perceived fiscal policy $\mathcal{M}^{PFP}$(i.e.,(24), (25), (26) and (27)) connects ξ_t with realizations of fiscal instruments $G_t, Z_t, \tau_{K,t}, \tau_{L,t}, B_t$ and output Y_t.

Define $\Omega_t = [\Omega_{G,t}, \Omega_{Z,t}, \Omega_{K,t}, \Omega_{L,t}]'$ where

$$\begin{aligned}
\Omega_{G,t} &= \log(G_t) + \rho_{G,Y}\log(Y_t) + \rho_{G,B}\log(B_t),\\
\Omega_{Z,t} &= \log(Z_t) + \rho_{Z,Y}\log(Y_t) + \rho_{Z,B}\log(B_t),\\
\Omega_{K,t} &= \log(\tau_{K,t}) - \rho_{K,Y}\log(Y_t) - \rho_{K,B}\log(B_t),\\
\Omega_{L,t} &= \log(\tau_{L,t}) - \rho_{L,Y}\log(Y_t) - \rho_{L,B}\log(B_t);
\end{aligned}$$

Ω_t are observables in the household's learning problem. The observable equation for the Kalman filter is given by

$$\underbrace{\begin{bmatrix}\Omega_{G,t}\\ \Omega_{Z,t}\\ \Omega_{K,t}\\ \Omega_{Z,t}\end{bmatrix}}_{\Omega_t} = \underbrace{\begin{bmatrix}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&\phi_{K,L}\\0&0&0&1&0&0&\phi_{K,L}&1\end{bmatrix}}_{H'}\underbrace{\begin{bmatrix}\Gamma_{G,t}\\ \Gamma_{Z,t}\\ \Gamma_{K,t}\\ \Gamma_{L,t}\\ u_{G,t}\\ u_{Z,t}\\ u_{K,t}\\ u_{L,t}\end{bmatrix}}_{\xi_t}$$

Define Q to be covariance matrix of v_t in the household's state equation. Q captures how households think $\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}$ are time-varying. Since

$\epsilon_{G,t}, \epsilon_{Z,t}, \epsilon_{K,t}, \epsilon_{L,t}$ are mutually $i.i.d.$ and $\eta_{X,t}$ are assumed to be mutually independent and are independent of the fiscal innovations $\{\epsilon_{X,t}\}$, the covariance matrix Q is diagonal. This establishes the household's learning problem as a linear state-space model as desired.

B.6 Algorithm of solving the RBC model

The algorithm follows Hollmayr and Matthes (2015) and Cogley et al. (2015) and can be separated into three stages. During the first stage households form expectations on endogenous variables by solving their perceived model. Let $\xi_{t|t-1}$ be the household's one period forecast of the policy rule coefficients $\Gamma_{X,t}$ and the AR(1) components $u_{X,t}$ at the beginning of time t[27], there is a one-to-one mapping $\mathcal{H}$ which links $\xi_{t|t-1}$ to the household's perceived steady states of the economy $\mathcal{SS}^{\circledast}_{t|t-1}$,

$$\mathcal{SS}^{\circledast}_{t|t-1} = \mathcal{H}(\xi_{t|t-1}) \tag{36}$$

Households then solve their perceived model $\{\mathcal{M}^{FP}; \mathcal{M}^{PFP}\}$ by log-linearizing around $\mathcal{SS}^{\circledast}_{t|t-1}$. In particular, let the vector of all endogenous variables(plus a constant intercept) in this economy be denoted Y_t, households solve the following perceived model

$$A(\mathcal{SS}^{\circledast}_{t|t-1})Y_t = B(\mathcal{SS}^{\circledast}_{t|t-1})E^{\circledast}_t Y_{t+1} + C(\mathcal{SS}^{\circledast}_{t|t-1})Y_{t-1} + D\epsilon^{\circledast}_t \tag{37}$$

The log-linearized difference equation is solved using the Sims (2002) algorithm. The solution is the household's perceived law of motion for the economy and is given by

$$Y_t = G(\mathcal{SS}^{\circledast}_{t|t-1})Y_{t-1} + M(\mathcal{SS}^{\circledast}_{t|t-1})\epsilon^{\circledast}_t$$

where $G(\mathcal{SS}^{\circledast}_{t|t-1})$ solves the following matrix quadratic equation

$$G(\mathcal{SS}^{\circledast}_{t|t-1}) = [A(\mathcal{SS}^{\circledast}_{t|t-1}) - B(\mathcal{SS}^{\circledast}_{t|t-1})G(\mathcal{SS}^{\circledast}_{t|t-1})]^{-1}C(\mathcal{SS}^{\circledast}_{t|t-1}) \tag{38}$$

and $M(\mathcal{SS}^{\circledast}_{t|t-1})$ is given by

$$M(\mathcal{SS}^{\circledast}_{t|t-1}) = [A(\mathcal{SS}^{\circledast}_{t|t-1}) - B(\mathcal{SS}^{\circledast}_{t|t-1})G(\mathcal{SS}^{\circledast}_{t|t-1})]^{-1}D. \tag{39}$$

[27] which is lastly updated at the end of $t-1$ after observing the realizations of Ω_{t-1};

During the second stage household's perceived law of motion drives the actual law of motion of the economy: this can be achieved by replacing the estimated policy coefficients $\Gamma_{X,t}$ in $C(\mathcal{SS}^{\circledast}_{t|t-1})$ of (37) with the true policy coefficients. Let the matrix be $C^{true}(\mathcal{SS}^{\circledast}_{t|t-1})$. The actual law of motion solves

$$A(\mathcal{SS}^{\circledast}_{t|t-1})Y_t = B(\mathcal{SS}^{\circledast}_{t|t-1})E^{\circledast}_t Y_{t+1} + C^{true}(\mathcal{SS}^{\circledast}_{t|t-1})Y_{t-1} + D\epsilon_t \tag{40}$$

where the household's perceived shocks $\epsilon^{\circledast}_t$ have been replaced by the actual shocks ϵ_t. Solution of (40) is given by

$$Y_t = H(\mathcal{SS}^{\circledast}_{t|t-1})Y_{t-1} + M(\mathcal{SS}^{\circledast}_{t|t-1})\epsilon_t. \tag{41}$$

where $M(\mathcal{SS}^{\circledast}_{t|t-1})$ is given by (39). $H(\mathcal{SS}^{\circledast}_{t|t-1})$ is given by

$$\begin{gathered} H(\mathcal{SS}^{\circledast}_{t|t-1}) = G(\mathcal{SS}^{\circledast}_{t|t-1}) + \\ [A(\mathcal{SS}^{\circledast}_{t|t-1}) - B(\mathcal{SS}^{\circledast}_{t|t-1})G(\mathcal{SS}^{\circledast}_{t|t-1})]^{-1}(C^{true}(\mathcal{SS}^{\circledast}_{t|t-1}) - C(\mathcal{SS}^{\circledast}_{t|t-1})) \end{gathered} \tag{42}$$

with $G(\mathcal{SS}^{\circledast}_{t|t-1})$ given by (38). Y_t summarizes the equilibrium dynamics of the economy at time t. Extracting realizations of fiscal instruments and output(in logs) from Y_t, households form Ω_t defined in section (B.5) to update their beliefs $(\xi_{t+1|t}, P_{t+1|t})$,

$$\begin{aligned} \xi_{t+1|t} &= F\xi_{t|t-1} + FP_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}(\Omega_t - H'\xi_{t|t-1}), \\ P_{t+1|t} &= F[P_{t|t-1} - P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}]F' + Q. \end{aligned}$$

At time $t+1$ the three stages can be repeated to solve the RBC model recursively. This completes the algorithm for solving the RBC model.

C Appendix C.

C.1 Linearized IIAU model for estimation

In this section I derive a linearized state transition equation for the underlying non-linear IIAU model. Along with a linear measurement equation, this linear state transition approximation allows me to apply standard Kalman filters to estimate deep structural parameters. I test the performance of this

linear approximation by running simulations and comparing solutions between the linearized and the non-linear IIAU models. Direct parameter estimation (either by the frequentist approach or the Bayesian approach) of the non-linear state-space model(SSM) by sequential Monte Carlo methods(i.e., particle filters) are computationally intensive. The good performance of the above linearization opens up possibilities of estimating posterior densities of structural parameters both effectively and efficiently.

I start by considering the steady-state Kalman filter (31), (32). This implies $P_{t+1|t} \to P$ where the steady-state covariance matrix P solves the following discrete-time algebraic Riccati equation

$$P = F[P - PH(H'PH + R)^{-1}H'P]F' + Q \tag{43}$$

and $\xi_{t+1|t}$ evolves according to

$$\xi_{t+1|t} = F\xi_{t|t-1} + FPH(H'PH + R)^{-1}(\Omega_t - H'\xi_{t|t-1}) \tag{44}$$

Define $K = FPH(H'PH + R)^{-1}$, K is the steady-state Kalman gain matrix (of size 8×4). Define $\tilde{\xi}_t = \xi_{t+1|t}$, (44) implies

$$\tilde{\xi}_{t+1} = (F - KH')\tilde{\xi}_t + K\Omega_{t+1}. \tag{45}$$

Recall the *true data generating process* of Ω_{t+1} is given by

$$\begin{bmatrix} \Omega_{G,t+1} \\ \Omega_{Z,t+1} \\ \Omega_{K,t+1} \\ \Omega_{L,t+1} \end{bmatrix} = H' \begin{bmatrix} \Gamma_G^* \\ \Gamma_Z^* \\ \Gamma_K^* \\ \Gamma_L^* \\ u_{G,t+1} \\ u_{Z,t+1} \\ u_{K,t+1} \\ u_{L,t+1} \end{bmatrix} = H' \begin{bmatrix} \Gamma_G^* \\ \Gamma_Z^* \\ \Gamma_K^* \\ \Gamma_L^* \\ \rho_G u_{G,t} + \sigma_G \epsilon_{G,t+1} \\ \rho_Z u_{Z,t} + \sigma_Z \epsilon_{Z,t+1} \\ \rho_K u_{K,t} + \sigma_K \epsilon_{K,t+1} \\ \rho_L u_{L,t} + \sigma_L \epsilon_{L,t+1} \end{bmatrix}$$

(45) thus says

$$\begin{bmatrix} \tilde{\Gamma}_{G,t+1} \\ \tilde{\Gamma}_{Z,t+1} \\ \tilde{\Gamma}_{K,t+1} \\ \tilde{\Gamma}_{L,t+1} \\ \tilde{u}_{G,t+1} \\ \tilde{u}_{Z,t+1} \\ \tilde{u}_{K,t+1} \\ \tilde{u}_{L,t+1} \end{bmatrix} = (F - KH') \begin{bmatrix} \tilde{\Gamma}_{G,t} \\ \tilde{\Gamma}_{Z,t} \\ \tilde{\Gamma}_{K,t} \\ \tilde{\Gamma}_{L,t} \\ \tilde{u}_{G,t} \\ \tilde{u}_{Z,t} \\ \tilde{u}_{K,t} \\ \tilde{u}_{L,t} \end{bmatrix} + KH' \begin{bmatrix} \Gamma_G^* \\ \Gamma_Z^* \\ \Gamma_K^* \\ \Gamma_L^* \\ \rho_G u_{G,t} + \sigma_G \epsilon_{G,t+1} \\ \rho_Z u_{Z,t} + \sigma_Z \epsilon_{Z,t+1} \\ \rho_K u_{K,t} + \sigma_K \epsilon_{K,t+1} \\ \rho_L u_{L,t} + \sigma_L \epsilon_{L,t+1} \end{bmatrix} \tag{46}$$

It is important to notice that $\tilde{u}_{X,t}$ and $u_{X,t}$ for $X = G, Z, K, L$ are two different processes. In particular, $u_{X,t}$ are exogenous AR(1) processes while $\tilde{u}_{X,t} = u_{X,t+1|t}$ is the household's perceived $u_{X,t+1}$, conditional on all available information at time t. $\tilde{u}_{X,t}$ is part of the output of household's Kalman filter and in general will not be a $AR(1)$ in $\epsilon_{X,t}$.

One important fact of (46) is that it's linear in states $[\tilde{\xi}_t, u_{G,t}, u_{Z,t}, u_{K,t}, u_{L,t}]$. Rewriting (46) as

$$\begin{bmatrix} \tilde{\Gamma}_{G,t+1} - \Gamma_G^* \\ \tilde{\Gamma}_{Z,t+1} - \Gamma_Z^* \\ \tilde{\Gamma}_{K,t+1} - \Gamma_K^* \\ \tilde{\Gamma}_{L,t+1} - \Gamma_L^* \\ \tilde{u}_{G,t+1} \\ \tilde{u}_{Z,t+1} \\ \tilde{u}_{K,t+1} \\ \tilde{u}_{L,t+1} \end{bmatrix} = (F - KH') \begin{bmatrix} \tilde{\Gamma}_{G,t} - \Gamma_G^* \\ \tilde{\Gamma}_{Z,t} - \Gamma_Z^* \\ \tilde{\Gamma}_{K,t} - \Gamma_K^* \\ \tilde{\Gamma}_{L,t} - \Gamma_L^* \\ \tilde{u}_{G,t} \\ \tilde{u}_{Z,t} \\ \tilde{u}_{K,t} \\ \tilde{u}_{L,t} \end{bmatrix} + KH' \begin{bmatrix} 0_{4\times 4} \\ I_4 \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} \rho_G & 0 & 0 & 0 \\ 0 & \rho_Z & 0 & 0 \\ 0 & 0 & \rho_K & 0 \\ 0 & 0 & 0 & \rho_L \end{bmatrix} \begin{bmatrix} u_{G,t} \\ u_{Z,t} \\ u_{K,t} \\ u_{L,t} \end{bmatrix} + KH' \begin{bmatrix} 0_{4\times 4} \\ I_4 \end{bmatrix} \begin{bmatrix} \sigma_G & 0 & 0 & 0 \\ 0 & \sigma_Z & 0 & 0 \\ 0 & 0 & \sigma_K & 0 \\ 0 & 0 & 0 & \sigma_L \end{bmatrix} \begin{bmatrix} \epsilon_{G,t+1} \\ \epsilon_{Z,t+1} \\ \epsilon_{K,t+1} \\ \epsilon_{L,t+1} \end{bmatrix}$$

As we shall see later, (47) will become part of the linearized state transition equation.

Let $\mathcal{SS}^{\circledast}_{t|t-1}$ denote households' perceived steady states of the economy at t, (36) in Appendix B.6 establishes a one-to-one mapping $\mathcal{H}$ between

$[\tilde{\Gamma}_{G,t}, \tilde{\Gamma}_{Z,t}, \tilde{\Gamma}_{K,t}, \tilde{\Gamma}_{L,t}]$ and $\mathcal{SS}^{\circledast}_{t+1|t}$ such that

$$\mathcal{SS}^{\circledast}_{t+1|t} = \mathcal{H}(\tilde{\Gamma}_{G,t}, \tilde{\Gamma}_{Z,t}, \tilde{\Gamma}_{K,t}, \tilde{\Gamma}_{L,t})$$

The actual law of motion of endogenous variables Y_t[28] is given by (41). I now linearize (41) around the full information, rational expectation(FIRE) steady states $\mathcal{SS}^*$(or, equivalently Y^*).

$$\begin{aligned} Y_{t+1} - Y^* &= H(\mathcal{SS}^{\circledast}_{t+1|t})Y + M(\mathcal{SS}^{\circledast}_{t+1|t})\epsilon_{t+1} - Y^* \\ &= H(\mathcal{SS}^{\circledast}_{t+1|t})(Y_t - Y^*) + M(\mathcal{SS}^{\circledast}_{t+1|t}) + H(\mathcal{SS}^{\circledast}_{t+1|t})Y^* - Y^* \\ \approx G(\mathcal{SS}^*)(Y_t - Y^*) &+ M(\mathcal{SS}^*)\epsilon_{t+1} + \frac{\partial H(\mathcal{SS}^{\circledast}_{t+1|t})Y^*}{\partial \tilde{\Gamma}_t{}'}|_{\tilde{\Gamma}_t = \Gamma^*}(\tilde{\Gamma}_t - \Gamma^*) \end{aligned} \quad (48)$$

where I utilize the fact that $H(\mathcal{SS}^{\circledast}_{t+1|t}) \to G(\mathcal{SS}^*), M(\mathcal{SS}^{\circledast}_{t+1|t}) \to M(\mathcal{SS}^*)$ as $\mathcal{SS}^{\circledast}_{t+1|t} \to \mathcal{SS}^*$ and matrices $G(\mathcal{SS}^*), M(\mathcal{SS}^*)$ solve the FIRE model. $\tilde{\Gamma} - \Gamma^*$ is the following 4×1 vector

$$\tilde{\Gamma} - \Gamma^* = \begin{bmatrix} \tilde{\Gamma}_{G,t} - \Gamma^*_G \\ \tilde{\Gamma}_{Z,t} - \Gamma^*_Z \\ \tilde{\Gamma}_{K,t} - \Gamma^*_K \\ \tilde{\Gamma}_{L,t} - \Gamma^*_L \end{bmatrix}$$

The Jacobian term

$$J = \frac{\partial H(\mathcal{SS}^{\circledast}_{t+1|t})Y^*}{\partial \tilde{\Gamma}_t{}'}|_{\tilde{\Gamma}_t = \Gamma^*} \quad (49)$$

in (48) will be calculated numerically by finite difference method, with $H(\mathcal{SS}^{\circledast}_{t+1|t})$ being output of (42) from Appendix B.6.

Combining (47) and (48) yields the state transition equation. The *expanded state variables* are given by

$$\{Y_t, \tilde{\Gamma}_{G,t} - \Gamma^*_G, \tilde{\Gamma}_{Z,t} - \Gamma^*_Z, \tilde{\Gamma}_{K,t} - \Gamma^*_K, \tilde{\Gamma}_{L,t} - \Gamma^*_L, \tilde{u}_{G,t}, \tilde{u}_{Z,t}, \tilde{u}_{K,t}, \tilde{u}_{L,t}\}$$

The *linearized state transition equation* can be written as

$$\begin{pmatrix} \tilde{\xi}_{t+1} \\ Y_{t+1} \end{pmatrix} = \begin{pmatrix} F - KH' & \mathbf{0} \;\; \Upsilon \\ J \;\; \mathbf{0} & G(\mathcal{SS}^*) \end{pmatrix} \begin{pmatrix} \tilde{\xi}_t \\ Y_t \end{pmatrix} + \begin{pmatrix} \mathbf{0} \;\; \Xi \\ M(\mathcal{SS}^*) \end{pmatrix} \epsilon_{t+1} \quad (50)$$

[28] Y_t is in logs and also contains a constant term 1;

J is the Jacobian matrix defined in (49) and

$$\Upsilon = KH' \begin{bmatrix} 0_{4\times4} \\ I_4 \end{bmatrix} \begin{bmatrix} \rho_G & 0 & 0 & 0 \\ 0 & \rho_Z & 0 & 0 \\ 0 & 0 & \rho_K & 0 \\ 0 & 0 & 0 & \rho_L \end{bmatrix}, \quad \Xi = KH' \begin{bmatrix} 0_{4\times4} \\ I_4 \end{bmatrix} \begin{bmatrix} \sigma_G & 0 & 0 & 0 \\ 0 & \sigma_Z & 0 & 0 \\ 0 & 0 & \sigma_K & 0 \\ 0 & 0 & 0 & \sigma_L \end{bmatrix};$$

In (50) Y_t are arranged such that the last four variables are $u_{G,t}, u_{Z,t}, u_{K,t}, u_{L,t}$.

C.2 Data construction

Construction of the data consumption C, investment I, government spending G, lump-sum transfers Z, capital and labor tax revenues TK, TL follows Leeper et al. (2010) and details can be found in Appendix B of the published paper. The remaining observables government debt B and hour worked L are constructed following Leeper et al. (2015). Details can be found in Section 3 of online appendix of the paper.

Unless otherwise stated, all data are from the Bureau of Economic Analysis's National Income and Product Accounts(NIPA). All data in levels are nominal values. GDP deflator for personal consumption expenditures (Table 1.1.4 line 2) is used to convert nominal data to real values. The following construction of C, I, G, TK, TL, Z, along with the descriptions, come from Leeper et al. (2010) directly. NIPA updated their dataset and added some new items. This causes the line numbers documented here are different than Leeper et al. (2010)'s. I keep these descriptions here for reference.

Consumption. Consumption, C, is defined as sum of personal consumption expenditures on nondurables (Table 1.1.5 line 5) and on services (Table 1.1.5 line 6).

Investment. Investment, I, is defined as sum of gross private domestic investment (Table 1.1.5 line 7) and personal consumption expenditures on durables (Table 1.1.5 line 4).

Government expenditure. Government expenditure, G, is defined as government consumption expenditure (Table 3.2 line 24), gross government investment (Table 3.2 line 44) , and net purchases of non-produced assets

(Table 3.2 line 46), minus government consumption of fixed capital (Table 3.2 line 47).

Capital income tax revenues. Capital income tax revenues, TK, is constructed by multiplying the average capital tax rate τ^K and its tax base. See Leeper et al. (2010) for details on how τ^K is constructed.

Labor income tax revenues. Labor income tax revenues, TL, is constructed by multiplying the average labor tax rate τ^L and its tax base. See Leeper et al. (2010) for details on how τ^L is constructed.

Lumpsum transfers. Lumpsum transfers, Z, is defined as net current transfers, net capital transfers, and subsidies (Table 3.2 line 35), minus the tax residue. Net current transfers are defined as current transfer payments(Table 3.2 line 25) minus current transfer receipts(Table 3.2 line 18). Net capital transfers are defined as capital transfer payments (Table 3.2 line 45) minus capital transfer receipts (Table 3.2 line 41). The tax residue is defined as current tax receipts(Table 3.2 line 2), contributions for government social insurance (Table 3.2 line 11), income receipts on assets (Table 3.2 line 14) and the current surplus of government enterprises (Table 3.2 line 22), minus total tax revenue, T (consumption, labor and capital tax revenues).

Government debt. Government debt, B, is the market value of privately held gross federal debt, obtained from the Federal Reserve Bank of Dallas. The quarterly values are constructed from the monthly values at the beginning of each quarter.

Hour worked. Hour worked, L, are defined as

$$L = \frac{H * Emp}{100}$$

where H is the index for non-farm business, all persons, average weekly hours duration, 2009 = 100, seasonally adjusted (from the U.S. Department of Labor), and Emp is the civilian employment for sixteen years and over, measured in thousands, seasonally adjusted (from U.S. Department of Labor). The Emp series is turned into an index where $2009Q3 = 100$.

C.3 Definition of observable variables

I first define output Y by adding up data $X = C, I, G$

$$Y = C + I + G$$

For $X = G, Z, B, TK, TL$, the variable-to-GDP ratios are defined as

$$GY = G/Y, ZY = Z/Y, BY = B/Y, TKY = TK/Y, TLY = TL/Y$$

The variable x is defined by making the following transformation to data X:

$$x = \ln\left(\frac{X}{Popindex}\right) * 100$$

where
Popindex: index of Pop, constructed so that $2009Q3 = 1$;
Pop: Civilian non-institutional population, ages 16 years and over, not seasonally adjusted. Number in thousands (from FRED, CNP16OV).
for $X = C, I, L$.
For $X = GY, ZY, BY, TKY, TLY$, the variable x is defined by

$$x = \ln(X) * 100.$$

c, i, l are independently linear detrended to yield $\tilde{c}, \tilde{i}, \tilde{l}$. I don't detrend gy, zy, by, tky and tly. Observables are linked to model variables in the following manner

$$\begin{bmatrix} \tilde{c}_t \\ \tilde{i}_t \\ \tilde{l}_t \\ gy_t \\ zy_t \\ by_t \\ tky_t \\ tly_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \times \ln(g^*/y^*) \\ 100 \times \ln(z^*/y^*) \\ 100 \times \ln(b^*/y^*) \\ 100 \times \ln(tk^*/y^*) \\ 100 \times \ln(tl^*/y^*) \end{bmatrix} + \begin{bmatrix} \hat{c}_t \\ \hat{i}_t \\ \hat{l}_t \\ \hat{g}_t - \hat{y}_t \\ \hat{z}_t - \hat{y}_t \\ \hat{b}_t - \hat{y}_t \\ \hat{tk}_t - \hat{y}_t \\ \hat{tl}_t - \hat{y}_t \end{bmatrix}$$

where $\hat{x}_t$ stands for 100 times the log deviation of variable x from its deterministic steady states x^* for $x = c, i, l, y, g, z, b, tk, tl$.

D Appendix D.

D.1 A short introduction to SMC

Sequential Monte Carlo(SMC) methods are a class of simulation-based algorithms to solve optimal filtering problems for non-linear non-Gaussian state space models. For an excellent tutorial on particle filtering and smoothing, see Doucet and Johansen (2008). For a nice survey of SMC on economics and finance applications, see Creal (2012).

Let $\mathbf{S}_t$ be the state variables at time t, $\mathbf{Y}_t$ is the measurement at time t, ϵ_t is the shock process, e_t is the measurement error, the general state-space model is given by

$$\begin{aligned}\mathbf{S}_{t+1} &= f(\mathbf{S}_t, \epsilon_{t+1})\\ \mathbf{Y}_t &= h(\mathbf{S}_t, e_t)\end{aligned}$$

where f, h are two arbitrary non-linear functions. The two densities p_{ϵ_t}, p_{e_t} are assumed to be known. Following Doucet and Johansen (2008) and Creal (2012), there are two general approaches of SMC methods to conduct the filtering problem of interest: a joint smoothing recursion approach and a marginal prediction and filtering recursion approach. The first approach works directly with the joint posterior $p(\mathbf{S_{0:t}}|\mathbf{Y_{1:t}};\theta)$ and utilize the following recursion

$$\begin{aligned}p(\mathbf{S_{0:t}}|\mathbf{Y_{1:t}};\theta) = &\ p(\mathbf{S_{0:t-1}}|\mathbf{Y_{1:t-1}};\theta)\frac{f(\mathbf{S}_t|\mathbf{S}_{t-1};\theta)h(\mathbf{Y}_t|\mathbf{S}_t;\theta)}{p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta)}\\ \propto &\ p(\mathbf{S_{0:t-1}}|\mathbf{Y_{1:t-1}};\theta)f(\mathbf{S}_t|\mathbf{S}_{t-1};\theta)h(\mathbf{Y}_t|\mathbf{S}_t;\theta)\end{aligned}$$

where th normalization constant is given by the contribution to the likelihood

$$p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta) = \int p(\mathbf{S_{0:t-1}}|\mathbf{Y_{1:t-1}};\theta)f(\mathbf{S}_t|\mathbf{S}_{t-1};\theta)h(\mathbf{Y}_t|\mathbf{S}_t;\theta)d\mathbf{S}_{0:t}$$

In consequence, the marginal distributions $p(\mathbf{S}_t|\mathbf{Y}_{1:t};\theta)$ can be obtained as a by-product by marginalizing out $\mathbf{S}_{0:t-1}$

$$p(\mathbf{S}_t|\mathbf{Y}_{1:t};\theta) = \int p(\mathbf{S}_{0:t}|\mathbf{Y}_{1:t};\theta)d\mathbf{S}_{0:t-1}$$

The second approach works directly with the marginal distribution $p(\mathbf{S}_t|\mathbf{Y}_{1:t};\theta)$ and consists of following recursive filtering and prediction steps

$$p(\mathbf{S}_t|\mathbf{Y}_{1:t};\theta) = \frac{h(\mathbf{Y}_t|\mathbf{S}_t;\theta)p(\mathbf{S}_t|\mathbf{Y}_{1:t-1};\theta)}{p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta)}$$

where

$$p(\mathbf{S}_t|\mathbf{Y}_{1:t-1};\theta) = \int f(\mathbf{S}_t|\mathbf{S}_{t-1};\theta)p(\mathbf{S}_{t-1}|\mathbf{Y}_{1:t-1};\theta)d\mathbf{S}_{t-1}$$

This marginal approach thus requires calculating the above integral and evaluating the contribution to the likelihood $p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta)$ in an on-line fashion[29],which is usually hard to do in practice.

D.2 The Schon et al. (2005) RBPF

The RBPF developed by Schon et al. (2005) is a mixture of the above two general approaches. Schon et al. (2005) also calls this RBPF a marginalized particle filter(MPF). Partitioning the state vectors as

$$\mathbf{S}_t = \begin{bmatrix} \mathbf{S}_t^l \\ \mathbf{S}_t^n \end{bmatrix},$$

this RBPF can handle models with the following structure

$$\begin{aligned} \mathbf{S}_{t+1}^n &= f_t^n(\mathbf{S}_t^n) + A_t^n(\mathbf{S}_t^n)\mathbf{S}_t^l + G_t^n(\mathbf{S}_t^n)w_t^n && (51) \\ \mathbf{S}_{t+1}^l &= f_t^l(\mathbf{S}_t^n) + A_t^l(\mathbf{S}_t^n)\mathbf{S}_t^l + G_t^l(\mathbf{S}_t^n)w_t^l && (52) \\ y_t &= h_t(\mathbf{S}_t^n) + C_t(\mathbf{S}_t^n)\mathbf{S}_t^l + e_t && (53) \end{aligned}$$

where the state noise is assumed white and Gaussian distributed with

$$w_t = \begin{bmatrix} w_t^l \\ w_t^n \end{bmatrix} \sim N(0, Q_t), \quad Q_t = \begin{bmatrix} Q_t^l & Q_t^{ln} \\ (Q_t^{ln})' & Q_t^n \end{bmatrix}.$$

[29] $p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta)$ can be calculated using the following integral

$$p(\mathbf{Y}_t|\mathbf{Y}_{1:t-1};\theta) = \int p(\mathbf{Y}_t|\mathbf{S}_t;\theta)p(\mathbf{S}_t|\mathbf{Y}_{1:n-1};\theta)d\mathbf{S}_t;$$

And the measurement error is white and Gaussian distributed according to

$$e_t \sim N(0, R_t).$$

$\mathbf{S}_t^l$ is assumed Gaussian, $\mathbf{S}_0^l \sim N(\overline{\mathbf{S}}_0, \overline{P}_0)$. The density of $\mathbf{S}_0^n$ can be arbitrary but is assumed known.

One crucial characteristics of the above state-space model is that conditional on non-linear states $\mathbf{S}_t^n$, the remaining model has a linear sub-structure in states $\mathbf{S}_t^l$. Kalman filter thus can be utilized to marginalize linear states $\mathbf{S}_t^l$ while particles only need to occupy a lower dimensional space. Such particle filters are called Rao-Blackwellized particle filters(RBPF) and is a direct application of the Rao-Blackwell Theorem. The Rao-Blackwell theorem states that if $g(X)$ is any kind of estimator of a parameter θ, then the conditional expectation of $g(X)$ given $T(X)$, where $T(X)$ is a sufficient statistics of θ, is typically a better estimator of θ.

More explicitly, the MPF considers the mixed marginal-joint posterior density and uses Bayes' theorem to give

$$p(\mathbf{S}_t^l, \mathbf{S}_{0:t}^n | \mathbf{Y}_{1:t}; \theta) = p(\mathbf{S}_t^l | \mathbf{S}_{0:t}^n, \mathbf{Y}_{1:t}; \theta) p(\mathbf{S}_{0:t} | \mathbf{Y}_{1:t}; \theta)$$

where the marginal component $p(\mathbf{S}_t^l | \mathbf{S}_{0:t}^n, \mathbf{Y}_{1:T}; \theta)$ is analytically tractable by Kalman filter and the joint component $p(\mathbf{S}_{0:t} | \mathbf{Y}_{1:t}; \theta)$ can be simulated by standard particle filters

$$p(\mathbf{S}_{0:t} | \mathbf{Y}_{1:t}; \theta) = p(\mathbf{S}_{0:t-1} | \mathbf{Y}_{1:t-1}; \theta) \frac{p(\mathbf{Y}_t | \mathbf{S}_{0:t}^n, \mathbf{Y}_{1:t-1}; \theta) p(\mathbf{S}_t^n | \mathbf{S}_{0:t-1}^n, \mathbf{Y}_{1:t-1}; \theta)}{p(\mathbf{Y}_t | \mathbf{Y}_{1:t-1}; \theta)}$$

Marginal distributions $p(\mathbf{S}_t | \mathbf{Y}_{1:t}; \theta)$, $p(\mathbf{S}_t^l | \mathbf{Y}_{1:t}; \theta)$ or $p(\mathbf{S}_t^n | \mathbf{Y}_{1:t}; \theta)$ are then obtained easily by Monte Carlo integrations. See *Algorithm 1* of Schon et al. (2005) for details of this MPF.

To apply the RBPF to the IIAU model, the crux is to recognize the non-linear state variables are $[\Gamma_{G,t}, \Gamma_{Z,t}, \Gamma_{K,t}, \Gamma_{L,t}]$ and the linear state variables are given by all endogenous economic variables Y_t, along with households' perceived AR(1) processes $[\tilde{u}_{G,t}, \tilde{u}_{Z,t}, \tilde{u}_{K,t}, \tilde{u}_{L,t}]$. The non-linear state transition equation (51) will be given as the output of households' Kalman filter while the linear station transition equation (52) comes from the output of the households' Kalman filter.