Dynamic Savings Choices with Wealth-Dependent Time Inconsistency∗

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We study a dynamic savings game where decision makers rotate in and out of power and value consumption more highly when in power, disagreeing on the utility function. We offer two interpretations: a behavioral economics based one, featuring a single consumer with time-inconsistent preferences, and a political economy based one, where alternating parties value private benefits from spending while they are in power. When disagreement is uniform, so that utility functions are proportional to one another, our setup is equivalent to hyperbolic discounting. We consider more general specifications to allow the degree of disagreement to vary with the level of spending. We focus on Markov equilibria. As is well known, dynamic games with hyperbolic discounting may be ill behaved. Our first contribution is technical, showing that this is not the case for our continuous-time model, proving existence by construction and characterizing equilibria. Our second contribution is to show that when the time-inconsistency problem varies with spending relatively rich dynamics may emerge. In particular, when disagreement is strongest for low spending poverty traps emerge: those with high enough wealth are able to save, but the rest become trapped in poverty. In contrast, when disagreement is strongest for high spending there is convergence to a unique steady state starting from any initial wealth level.

1 Introduction

Temptations or time-inconsistency problems help explain a number of phenomena and have received ample attention from the economics literature. However, the extent of these problems likely varies quite a bit according to the situation. For example, the self-control problems to save faced by a rich person, say, tempted to use wealth to buy a yacht, are

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quite different from those faced by a relatively poor person, tempted to spend the remain-
der of their paycheck on a night out with friends. Similarly, for political economy rea-
sons, governments may behave in a present-biased manner, but it also seems likely that
these problems may be quite different for advanced countries than for developing coun-
tries. The strength of time-inconsistency problems may depend on the level of wealth or
spending. This possibility has, comparatively, received relatively little attention from the
literature.

In this paper we study a flexible dynamic game of savings. Decision makers rotate in
and out of power and disagree on the utility function. Those in power value consumption
more highly, leading to a time-inconsistency problem. The difference in utility functions
may vary with the level of spending, affecting the concentration and location of the time
inconsistency problem.

Our model can be interpreted in at least two ways. First, following Phelps and Pol-
lak (1968) and Laibson (1997), the model may describe the problem of a single consumer
playing an intertemporal game against future ‘selves’. The disagreement on the utility
function generates a time inconsistency problem that is similar, but strictly generalizes,
hyperbolic discounting. Indeed, our model specification adapts the one proposed by Har-
ris and Laibson (2013) which adopts this behavioral interpretation.

Second, one can interpret our model as describing a political economy game where
each decision maker represents a party and the governing party has the power to control
the budget. In this case, the ruling party may value private benefits from spending while
in power, such as pork spending or outright transfers to party members, leading to a time
inconsistency problem. This interpretation relates our work to the literature on political
economy models of debt, such as Alesina and Tabellini (1990), Amador (2002), Battaglini
and Coate (2008) and others, who study political economy dynamic games of debt.

For both the behavioral and political economy applications, our results uncovers some
interesting new possibilities for the long-run dynamics, especially in the possibility of
poverty traps.

A distinguishing feature of our approach is to consider relatively flexible forms of dis-
agreements on the utility function. This gives rise to a time-inconsistency problem that is
not uniform and instead varies with the level of spending and wealth. In contrast, most
of the literature adopts variations of a hyperbolic discounting setting where the time in-
consistency problem is uniform. Indeed, a popular and simple case in the hyperbolic
discounting literature that helps underscore this point is obtained when utility is a power
function and no binding borrowing constraints are imposed. A linear Markov equilib-
rium then exists, fully characterized by a constant saving rate, that applies at all wealth
levels. The time inconsistency problem leads to a uniformly lower savings rate (Laibson, 1997).

In contrast, in our model, the time inconsistency problem may not be uniform. In the special case where the utility functions in and out of power are proportional to each other, our setup does collapses to the hyperbolic discounting case. However, we consider more general forms of disagreement, ones that vary with the level of spending. In this way, our approach shares the spirit of Banerjee and Mullainathan (2010).\(^1\) The main difference is that they limit their analysis to two- or three-period models, whereas we study a dynamic infinite-horizon continuous-time model that allows the strategic interactions to fully play out. An infinite horizon is crucial for our goal of deriving the long-run dynamics of assets.\(^2\)

Our focus is on Markov equilibria for our dynamic game.\(^3\) This the standard refinement adopted in the behavioral and political economy literatures and serves to focus on a situation without commitment, avoiding implicit commitment mechanisms through reputation.

Our first contribution is technical. As is well known, dynamic saving games may be ill behaved. In particular, Krusell and Smith (2003) proves that the basic model features a continuum of Markov equilibria with discontinuous policy functions. There are many ways to deal with these issues. For example, Chatterjee and Eyigungor (2014) introduces lotteries that smooth out the solution. Harris and Laibson (2013) introduced a continuous-time model and focused on limit case with ‘instant gratification’. Our approach is closest to Harris and Laibson (2013), although we do not impose the instant gratification limit. We show that our model is tractable away from this limit and prove the existence of well-behaved equilibria in our continuous-time model. Indeed, our proof of existence works by construction, allowing us to characterize the equilibria as a by product, while also providing a convenient numerical procedure for applications.

Our second contribution is substantive, since we characterize the equilibrium sharply, deriving its long run implications for assets. We show that when time inconsistency does not vary sufficiently, the hyperbolic discounting model being a special case, the equilib-

\(^1\) Alesina and Tabellini (1990) also consider a relatively general form of disagreement on the composition of spending (see their equations 1). However, for their analysis, they specialize to corner cases, a more extreme and uniform disagreement (see their equations 4 and 5).

\(^2\) One important contribution of Banerjee and Mullainathan (2010) is to motivate and derive the time-inconsistency problem from the allocation of spending across many goods with separable utilities and disagreements on these utilities. We adopt more directly a disagreement on the utility over spending, although we also briefly review a motivation for the differences in utilities based on many goods.

\(^3\) Bernheim et al. (2014) consider sub-game perfect equilibria for the hyperbolic discounting game and characterize the dynamics of asset under different equilibria.
rium features either global saving or dissaving.

However, we show that when the time inconsistency problem varies with spending then rich dynamics may emerge, with saving or dissaving coexisting at different asset levels. In particular, if the time inconsistency problem is stronger at low spending levels, poverty traps may emerge. If instead the time-inconsistency problem it is stronger when spending is high, then a stable interior steady state may exist.

2 A Dynamic Savings Game

A sequence of decision makers rotate in and out of power.\textsuperscript{4} The agent in power controls consumption and savings and anticipates remaining in power for a stochastic interval of time, losing power at a Poisson rate to its successor. Once removed, this agent continues to care about the future consumption path. However, since the agent’s marginal utility from consumption is higher while in power, this creates a time-inconsistency problem. As a result, one must approach the problem as a dynamic game.\textsuperscript{5}

Our model is cast in continuous time. This is crucial to our approach, techniques and results. It allows us to construct equilibria as solutions to differential equations. Intuitively, since wealth evolves continuously over time, this allows us to characterize behavior locally. The stochastic alternation of power helps smooth out the time inconsistency problem and is borrowed from \textit{Harris and Laibson (2013)}, extended to allow for more general differences in preferences. However, we will not focus on the ‘instant gratification’ limit (as the Poisson rate for losing power goes to infinity) as they did, so our strategy for characterizing things is quite different.

Next, we introducing the details of the model environment. We then offer a few interpretations and special cases.

2.1 Environment

Time is continuous with an infinite horizon, denoted by $t \in [0, \infty)$. We next describe the preferences and the constraints agents face.

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\textsuperscript{4}It is also possible to describe the economy with two agents, that rotate in and out of power, see the discussion in Subsection 3.1.

\textsuperscript{5}In contrast, when preferences are the same in and out of power, then the rotation of agents is of no consequence. Indeed, the equilibrium coincides with optimum for a dictator agent that always remains power—an application of the Principle of Optimality of Dynamic Programming.
Preferences. At each point in time, the utility function from consumption is given by

\[ U_{\theta_t}(c_t), \]

where \( \theta_t \in \{0, 1\} \) is an indicator with value \( \theta_t = 1 \) when the agent is in power and \( \theta_t = 0 \) otherwise. For each \( \theta \), we assume \( U_\theta : \mathbb{R}_+ \to \mathbb{R} \) is concave, increasing, continuous and differentiable. In addition, we assume \( U_1 \) satisfies the INADA conditions \( U'_1(0) = \infty \) and \( \lim_{c \to \infty} U'_1(c) = 0 \).

Agents in power are removed at a constant Poisson arrival rate \( \lambda \geq 0 \). Thus, tenure is stochastic with average length \( \lambda^{-1} \). To simplify we assume power can never be regained, so that \( \theta_t = 0 \) is an absorbing state for each agent; however, we later show that an economy where power may be regained at some Poisson rate can be transformed to yield the same equilibrium conditions that we study. This Poisson specification follows Harris and Laibson (2013) closely, although they focus their analysis on the case with \( \lambda \to \infty \), whereas we will consider finite \( \lambda \).

The continuation lifetime utility at time \( t \) for an agent in power is

\[ V_t \equiv \mathbb{E}_t \left[ \int_0^\tau e^{-\rho s} U_1(c_{t+s})ds + e^{-\rho \tau} W_t \right], \quad (1) \]

where \( \rho > 0 \) is a discount rate and \( \tau \) is the random time at which the agent currently in power is removed, distributed according to the c.d.f. \( 1 - e^{-\lambda \tau} \). Here \( W_t \) represents the the continuation lifetime utility once out of power,

\[ W_t \equiv \int_0^\infty e^{-\rho s} U_0(c_{t+s})ds. \quad (2) \]

The expectations operator in these expressions averages over the only underlying shock in the economy, the alternation of power. In principle, consumption \( c_t \) could also be stochastic. However, given the symmetry of preferences and our focus on Markov equilibria, the path for consumption will be deterministic.

Disagreement between those in and out of power captured by the difference in the utility functions \( U_0 \) and \( U_1 \), is crucial to our model. Define the ratio of marginal utilities

\[ \beta(c) \equiv \frac{U'_0(c)}{U'_1(c)}. \]

\(^6\)The concavity and differentiability of \( U_0 \) is not crucial but simplifies the analysis. Indeed, an earlier version of this paper focused on a case that had a convex kink in \( U_0(c) \).
The ratio $\beta(c)$ is a measure of local disagreement. We will consider different shapes for $\beta(c)$ which capture various patterns of disagreement or temptation. Throughout the paper we assume that the marginal utility from consumption is higher when in power, so that $\beta(c)$ is (weakly) less than one.

**Assumption 1 (Present Bias).** The utility functions $U_1$ and $U_0$ are such that for all $c > 0$

$$\beta(c) \in [\overline{\beta}, 1]$$

for some $\overline{\beta} > 0$.

When $\beta(c) < 1$, agents prefer to consume relatively more while in power, this leads to a present-bias time-inconsistency problem. Those out of power would like those in power to exercise restraint, lowering consumption and increasing savings. Those in power would like to commit their successors somehow, but have no way to do so.

**Budget Constraints and Borrowing Limits.** While in power, the agent chooses $c_t \geq 0$ and assets evolve according to the budget constraint

$$\dot{a}_t = ra_t + y - c_t,$$

where $a_t$ denotes the total asset wealth. The interest rate $r > 0$ and income $y \geq 0$ are exogenous and constant.

The agent is also subject to a borrowing constraint

$$a \leq a_t,$$

which requires that assets stay above a certain level, possibly negative. When $a = -\frac{y}{r} \leq 0$ this represents the so-called “natural” borrowing limit, that allows the agent to borrow against all future income.\(^7\) We are interested in situations where the borrowing constraint is tighter, thus we require

$$a > -\frac{y}{r}.$$  

Note that whenever $a_t = a$, we must have $\dot{a} \geq 0$ so that $c_t \leq y + ra$.

**Parameters and Normalizations.** We are particularly interested in situations with $r \geq \rho$ since this provides an incentive for the agent to save, but this incentive is potentially im-

\(^7\)The borrowing constraint can also be interpreted as a commitment device. In the case of consumers this may be forced savings and social security or illiquid assets. In the political-economy context, this could capture wealth funds, which limit discretionary spending from natural resources.
peded by the time inconsistency problem. In contrast, when \( r < \rho \) even a time-consistent agent, with \( \lambda = 0 \) or \( \beta(c) = 1 \), would not save. The present bias, with \( \lambda > 0 \) and \( \beta(c) < 1 \), only reinforces this tendency.

In fact, as we shall show, what turns out to be crucial is the value of \( \beta(c) \) relative to a benchmark given by

\[
\hat{\beta}(r, \rho, \lambda) \equiv \frac{\rho}{r} \left(1 - \frac{r - \rho}{\lambda}\right).
\]

When \( r > \rho \) then \( \hat{\beta}(r, \rho, \lambda) < 1 \). As we shall see, when \( \beta(c) < \hat{\beta}(r, \rho, \lambda) \) temptations are strong, providing a strong force for dissaving; this is always the case under Assumption 1 and \( r < \rho \). When \( \beta(c) > \hat{\beta}(r, \rho, \lambda) \) temptations are sufficiently weak, allowing positive savings; under Assumption 1, this requires \( r > \rho \).

To simplify the exposition we focus on the case without income, setting \( y = 0 \). This is without loss of generality because by a change of variables one can transform a problem with positive income \( y > 0 \) to one without: define \( \tilde{a}_t = a_t + \frac{y}{r} \) then \( \tilde{a}_t = r\tilde{a}_t - c_t \) and \( \tilde{a} \geq \tilde{a} \equiv a + \frac{y}{r} \). As a result of this transformation, our borrowing limit is transformed into a positive lower bound on assets: \( a > 0 \).

2.2 Interpretation and Special Cases

In this subsection we first discuss an important special case where disagreement is uniform, so that \( \beta(c) \) is constant. We then discuss one motivation for nonuniform disagreement. Finally, we provide a Generalized Euler equation that highlights intuitively the role of \( \beta(c) \) in saving decisions.

**Hyperbolic Discounting and Instantaneous Gratification.** A special and interesting case occurs when

\[
\beta(c) = \beta
\]  

for some \( \beta < 1 \), so that \( U_0(c) = \beta U_1(c) \). This corresponds to the continuous-time hyperbolic discounting model introduced by Harris and Laibson (2013), which in turn builds on the discrete-time treatments in Harris and Laibson (2001) Laibson (1997) and Phelps and Pollak (1968). They further specialize by considering the limit as

\[
\lambda \to \infty,
\]

the so-called ‘Instantaneous Gratification’ case. Tractability is gained in this limit from the fact that \( V_t \to W_t \), so that only a single continuation value is required. Harris and
Laibson (2013) also add uncertainty to the rate of return, considering the limit without noise to select equilibria. They provide a tight characterization.

Our paper considers general temptation shapes $\beta(c)$ and finite $\lambda < \infty$. We mainly focus on the standard setting without uncertainty, but introduce uncertainty in returns and income in Section .

**Many Goods, Disagreement and Engel Curves.** One interesting motivation for differences in $U_0$ and $U_1$ is to interpret $c$ as overall spending on various goods. Differences in $U_0$ and $U_1$ then arises from disagreement on how to spend across goods (Alesina and Tabellini, 1990; Banerjee and Mullainathan, 2010).

To see this, consider two goods, $x$ and $z$, with equal price normalized to unity. We assume these utility functions are additively separable in $x$ and $z$. Suppose good $x$ is valued equally by those in and out of power, but that $z$ is not valued at all by those out of power (less extreme assumptions work similarly). Thus, utility for those in power is $h(x) + g(z)$, for some utility functions $f$ and $g$, while it is simply $h(x)$ for those out of power.

At any point in time the agent in power solves a static subproblem: choosing how much to spend across $x$ and $z$ given total expenditure $c$. This defines indirect utility functions $U_1$ and $U_0$ as

$$U_1(c) = \max_{x+z=c} \{h(x) + g(z)\} \quad (6a)$$

$$U_0(c) = h(\hat{x}(c)), \quad (6b)$$

where $(\hat{x}(c), \hat{z}(c))$ denotes the solution to the maximization. The next result shows that we can generate any $U_1$ and $U_0$ in this way.

**Proposition 1.** Given $U_1$ and $U_0$ satisfying Assumption 1, there exists a strictly concave functions $h$ and $g$ so that (6) holds.

**Proof.** Appendix A.1.

Note that $U_0'(c) = h'(\hat{x}(c))$ and $U_1'(c) = h'(\hat{x}(c))\hat{x}'(c)$, implying

$$\beta(c) = \hat{x}'(c) = 1 - \hat{z}'(c) \leq 1.$$ 

A high marginal propensity to spend on $z$ lowers the marginal utility of spending for those out of power, since they do not value $z$. Thus, the shape of the Engel curve dictates the shape of the ratio $\beta(c)$. For example, when $\hat{z}(c)$ is concave, so that the marginal
propensity to spend on \( z \) is decreasing, the ratio \( \beta(c) \) is increasing. Intuitively, time inconsistency problems are then greater for relatively low levels of spending, where marginal increases in spending are allocated to the good that only those in power value.\(^8\)

**Generalized Euler Equation.** To illustrate the importance of the ratio \( \beta(c) \) we show in the Online Appendix that,

\[
\frac{d}{dt} \left( U'_1(c_t) \right) = (\rho - r) U'_1(c_t)
+ \lambda \int_0^{T^*} e^{\int_0^s (\rho + \lambda - r + \hat{c}'(a_{t+s})) \, ds_1} U'_1(c_{t+s}) (1 - \beta(c_{t+s})) \hat{c}'(a_{t+s}) \, ds,
+ \lambda e^{\int_0^{T^*} (\rho + \lambda - r + \hat{c}'(a_{t+s})) \, ds_1} U'_1(ra_{T^*}) (1 - \beta(ra_{T^*}))
\]

where \( \hat{c}(a) \) is the consumption decision of the decision maker as a function of total asset in a Markov equilibrium, and \( T^* \) the time that asset reaches a steady state, i.e. \( \hat{c}(a_{T^*}) = ra_{T^*} \); if \( T^* = \infty \) the last term is zero.\(^9\) This is a continuous-time version of what Harris and Laibson (2001) call a Generalized Euler equation. When \( \lambda = 0 \) or \( \beta(c) = 1 \) it reduces to the standard Euler equation, stating that the marginal utility \( U'_1(c_t) \) grows at rate \( \rho - r \).

However, when \( \lambda > 0, \beta(c) < 1 \) then as long as \( \hat{c}'(a) > 0 \) the second and third terms on the right hand side are nonzero and positive. Intuitively, if the agent spends today there is a nonzero probability that the additional assets will be used to increase consume once this agent is out of power. This extra consumption is positive as long as \( \hat{c}'(a) > 0 \) but it is valued less, since \( \beta(c) < 1 \). This contributes to a form of impatience. This effect can lead to a declining path for consumption and assets even when \( r > \rho \).

### 2.3 Full Commitment

An important benchmark is provided by the problem with commitment, where we imagine an agent currently in power that can choose a deterministic path for consumption \( \{c_t\} \) that is applied whether or not this agent remains in power.\(^{10}\) Lifetime utility with such a

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\(^8\)In the representation (6), we normalized the relative price of \( z \) to one, which is without loss of generality since we will not consider changes in this price. In general, changes in this price may affect \( \beta(c) \), for given \( h \) and \( g \), although a higher price may increase or decrease \( \beta \). Akerlof and Shiller (2015) discuss various other determinants of temptation, such as advertising.

\(^9\)In the Online Appendix, we provide an approximation for \( T^* \) when it is finite.

\(^{10}\)A still better commitment technology would allow consumption to be contingent on whether the agent is currently in power or not. However, deterministic consumption is more easily comparable to outcomes in our Markov equilibria.
commitment is given by

\[
V_{sp}(a) \equiv \max_{\{c_t\}} \mathcal{J} = \max_{\{c_t\}} \int_0^\infty \left( e^{-\lambda t} U_1(c_t) + (1 - e^{-\lambda t}) U_0(c_t) \right) dt
\]  

(7)

where the maximization is subject to \(a_0 = a, \dot{a}_t = ra_t - c_t\) and \(a_t \geq a\). Here \(e^{-\lambda t}\) represents the probability that the agent is still in power after time \(t\).\(^{11}\) Clearly, any equilibrium is worse than the commitment one for the agent in power, \(V(a) \leq V_{sp}(a)\).

Whenever \(r > \rho\) the commitment solution would eventually feature positive savings.

**Proposition 2.** If \(r > \rho\) the commitment solution eventually features positive savings and rising consumption, so that \(\dot{a}_t, \dot{c}_t > 0\) for all \(t > T\) for some \(T > 0\). Moreover, assets and consumption grow without bound, so that \(a_t, c_t \to \infty\) as \(t \to \infty\).

**Proof.** Appendix A.3.

This result is driven by the fact that as \(t \to \infty\) the probability of remaining in of power converges to zero. Eventually the commitment problem approaches a standard optimization problem with utility function \(U_0\) and, since \(r > \rho\), the solution then requires an increasing consumption path with asset accumulation. As we shall see, without commitment this is no longer the case and dissaving may emerge even when \(r > \rho\). This is one simple manifestation of the time-inconsistency problem.

### 3 Markov Equilibria

We focus on Markov equilibria with wealth \(a_t\) as the state variable. A Markov equilibrium consists of a policy function \(\hat{c}(a)\) for consumption as a function of wealth that maximizes the right hand side of (1), taking as given the value function \(W(a)\) defined by (2). We next develop these requirements more explicitly.

#### 3.1 Hamilton-Jacobi-Bellman Equations

We take advantage of the fact that time is continuous to represent a Markov equilibrium by the solution to differential equations. Consider the Hamilton-Jacobi-Bellman equations

\[11\] We derive (7) by integrating out, from the expected value

\[V_{sp} = \int_\tau^\infty \int_0^\infty e^{-\rho t} (1_{\{t \leq \tau\}} U_1(c_t) + 1_{\{t > \tau\}} U_0(c_t)) dt dF(\tau),\]

the Poisson uncertainty in the stopping time \(\tau\) at which the current decision maker is out of power. \(F(\tau) = 1 - e^{-\lambda \tau}\) is the CDF for \(\tau\).
\[ \rho V(a) = \max_{c \geq 0} \{ U_1(c) + V'(a)(ra - c) + \lambda(W(a) - V(a)) \}, \quad (8a) \]
\[ \rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra - \hat{c}(a)), \quad (8b) \]

where \( \hat{c}(a) \) denotes the solution to the maximization in (8a), which includes the additional constraint \( c \leq ra \) when \( a = a \). The implied dynamics for wealth is

\[ \dot{a}_t = ra_t - \hat{c}(a_t). \quad (9) \]

A Markov equilibrium is a pair of value functions \( (V(a), W(a)) \) and a consumption function \( \hat{c}(a) \) satisfying (8) and the following properties: (a) \( V \) is continuous and piecewise differentiable; (b) \( W \) is piecewise continuous and piecewise differentiable; (c) \( W \) is continuous and differentiable at any point where \( \hat{c}(a) \neq ra \); (d) \( W \) is continuous from the left at any point where \( ra < \hat{c}(a) \) in a neighborhood to the left of \( a \); (e) conversely, \( W \) is continuous from the right at any point where \( ra > \hat{c}(a) \) in a neighborhood to the right of \( a \); and (f) for any \( a \geq a_0 \) the ODE \( \dot{a}_t = ra_t - \hat{c}(a_t) \) with initial condition \( a_0 \) admits a solution \( \{a_t\}_{t=0}^{\infty} \) with \( a_t \geq a_0 \) for all \( t \geq 0 \) satisfying \( \lim_{t \to \infty} e^{-\rho t} V(a_t) = 0 \) and \( \lim_{t \to \infty} e^{-\rho t} W(a_t) = 0 \).

The conditions for a Markov equilibrium are relatively straightforward. The only subtle issue worth discussing here are the smoothness requirements for \( V \) and \( W \). The function \( V \) must be continuous, condition (a), because it represents the value from a continuous-time optimal control problem with a controllable state with continuous payoffs in the control.\(^{12}\) In contrast, the function \( W \) may be discontinuous, because it is not the value from a smooth optimization. However, since \( W(a_t) = \int_t^{\infty} e^{-\rho(s-t)} U_0(\hat{c}(a_s)) ds \), the function \( W \) is continuous and differentiable (differentiating, one recovers (8b)) along interval of assets that are connected by a path \( \{a(t)\} \). These considerations lead to conditions (b)–(e). Note that when the policy function \( \hat{c} \) is such that there are multiple disconnected ergodic sets, so that assets are not connected, then \( W \) potentially jumps at the boundaries of these sets.\(^{13,14}\) In the Online Appendix, we prove a Verification Theorem that if \( (V, W) \) is a solution to the system (8), then \( V \) is the value function of a decision maker that maximizes (1) subject to the budget constraint (3) and the borrowing con-
Constant Wealth and Consumption. It is useful to define the value of holding wealth and consumption constant,

\[ V(a) = \frac{1}{\rho + \lambda} U_1(ra) + \frac{\lambda}{(\rho + \lambda) \rho} U_0(ra) \]  

\[ W(a) = \frac{1}{\rho} U_0(ra). \]  

These value functions will play an important role in our constructions and proofs. The equilibrium value functions \((V, W)\) must coincide with \((\bar{V}, \bar{W})\) at steady states points, where \(\hat{c}(a) = ra\) so that \(\dot{a} = 0\).

### 3.2 Solution Technique

We now describe our technique for constructing Markov equilibria, which underlies the formal results as well as our numerical examples.

**Roots for \(V'(a)\).** Equation (8a) is an implicit differential equation, to be solved for \(V'(a)\) at each point \(a\), given values for \(V(a)\) and \(W(a)\). In the appendix we show that the right hand side of (8a) is a strictly convex function with a unique minimum, so that there are zero, one or two solutions (roots) for \(V'(a)\). The first-order condition for the maximization in (8a) gives

\[ U_1'(\hat{c}(a)) = V'(a). \]  

Thus, consumption is inversely related to \(V'(a)\). Indeed, when there exists two roots for \(V'(a)\), the lower root corresponds to dissaving, \(\hat{c}(a) > ra\) and \(\dot{a} < 0\), while the higher root corresponds to saving, \(\hat{c}(a) < ra\) and \(\dot{a} > 0\). When a unique root for \(V'(a)\) exists it is associated with \(\hat{c}(a) = ra\) and \(\dot{a} = 0\).

**Solving the ODEs.** The equilibrium analysis in the next sections involves solving the Hamilton-Jacobi-Bellman ODE equations (8) for the value functions \((V, W)\) and associated policy function \(\hat{c}(a)\) to satisfy the equilibrium requirements (a)–(f) mentioned above.

We construct equilibria by solving the ODEs starting at the bottom and working up; or by starting at the top and working down; or by combining both procedures. In more

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15The difficulty relative to the standard Verification Theorem (Fleming and Soner, 2005) is that \(W\) may be discontinuous and \(V\) is not differentiable at points of discontinuity of \(W\).
detail, the construction involves the following parts: (i) solving the ODEs with the appropriate root over an interval of assets; (ii) decide if and where to engineer a jump in \( W \) and the appropriate jump in \( W \); and (iii) imposing boundary conditions, either at the borrowing constraint or for high enough assets. Each one of the components is relatively straightforward. The great advantage of this technique is that part (i) is straightforward and local in nature. Part (ii) is aided by the fact that \( V \) must be continuous and that \( W \) can only jump to a self-generating value, i.e. consistent with (2), which reduces the possibilities greatly. Finally, the boundary conditions required for part (iii) are supplied at the extremes by known solutions: either getting absorbed by the borrowing constraint at the bottom or reaching a high enough level of assets where a known solution is available, such as the commitment solution.\(^{16}\)

**Dealing with Singularities at Steady States.** One technical challenge is that the differential system (8) is singular at steady states.\(^{17}\) At any point \( a_0 \) with \( V(a_0) = \overline{V}(a_0) \) and \( V(a_0) = \overline{W}(a_0) \) we have \( \hat{c}(a_0) = ra_0, \) so condition (8b) at \( a_0 \) cannot determine \( W'(a_0) \). As a result, we cannot apply standard existence theorems for regular ODEs.

The following lemma shows the existence of the solution \((V, W)\) locally around such points of singularity. Away from these points the system (8) is non-singular, so we can apply standard ODE methods to extend the solution.

**Lemma 1.** Suppose \( \beta(ra_0) < \hat{\beta}(r, \rho, \lambda) \). Then system (8) with initial condition \((V(a_0), W(a_0)) = (\overline{V}(a_0), \overline{W}(a_0))\) admits a solution over the interval \([a_0, a_0 + \omega]\) for some \( \omega > 0 \), with

1. \( V(a) > \overline{V}(a) \) for \( a > a_0 \);
2. \( \hat{c}(a) > ra \) for \( a > a_0 \), \( \lim_{a \downarrow a_0} \hat{c}(a) = ra_0 \) and \( \lim_{a \downarrow a_0} \hat{c}'(a) = \infty \).

**Proof.** The details of the proof are in Appendix C, here we provide a sketch. Consider initial values

\[ (V_c(a_0), W_c(a_0)) = (\overline{V}(a_0), \overline{W}(a_0) - \epsilon), \]

\(^{16}\)The equilibrium coincides with the commitment solution at a high enough level of asset if disagreement disappears at a high enough level of consumption: \( \frac{U_c'(c)}{U_c(c)} = 1 \) for \( c \geq \hat{c} \). Alternatively, a known solution can be found if \( U_0(c) = \beta U_1(c) \) for \( c \geq \hat{c} \) and \( U_1(c) \) is a power function for \( c \geq \hat{c} \). Another possibility is imposing an upper bound on assets. Basically, what we require is that there be a known solution for high enough assets, to serve as a boundary condition.

\(^{17}\)When we rewrite system (8) as a differential algebraic equation (DAE), the steady states correspond to critical singular points. However, the DAE at this point does not satisfy the sufficient conditions provided in the literature, for example in Rabier and Rheinboldt (2002), for the existence and uniqueness of solutions around singular points of DAEs, except for the case \( \lambda = 0 \).
where $\epsilon > 0$. With this boundary condition, the system is nonsingular at $a_0$, so we can find a solution over some interval $[a_0, a_0 + \omega]$ for some $\omega > 0$ that is independent of $\epsilon$. We then take the limit $\epsilon \to 0$ and show that the sequence of solutions converges to a well-defined limit that constitutes a solution to the original system with the desired properties.

Note that under this construction assets are falling, $\dot{a} < 0$ for $a > a_0$, towards the steady state at $a_0$. This implies that the values constructed are “self generating” in the sense that the interval $[a, a + \omega]$ is ergodic, forming a closed self-referential system.

### 3.3 Recovery of Power

We have assumed that an agent ousted from power cannot recover power. We now show that we can handle cases where power can be recovered at some Poisson arrival rate $\hat{\lambda} > 0$. The Hamilton-Jacobi-Bellman system becomes

\[
\begin{align*}
\rho V (a) &= \max_c U_1 (c) + V' (a) (ra - c) + \lambda (W (a) - V (a)) \\
\rho W (a) &= U_0 (\hat{c} (a)) + W' (a) (ra - \hat{c} (a)) + \hat{\lambda} (V (a) - W (a)).
\end{align*}
\]

If we define $\tilde{\lambda} \equiv \lambda + \hat{\lambda}$, $\tilde{W} \equiv \frac{\lambda}{\lambda + \hat{\lambda}} W + \frac{\hat{\lambda}}{\lambda + \hat{\lambda}} V$ and $\tilde{U}_0 (c) \equiv \frac{\lambda}{\lambda + \hat{\lambda}} U_0 (c) + \frac{\hat{\lambda}}{\lambda + \hat{\lambda}} U_1 (c)$, then one can show that we recover a system equivalent to (8a)–(8b) for $(V, \hat{W}, \hat{\lambda}, U_1, \hat{U}_0)$. This shows that the assumption that power cannot be recovered is without loss of generality.

### 4 Dissaving and Saving

In this section we construct and characterize Markov equilibria. We focus on situations where the ratio $\beta(c)$ is relatively stable. When $\beta(c)$ is below $\hat{\beta}(r, \rho, \lambda)$, we construct equilibria where the agent dissaves at all asset levels; when $\beta(c)$ is above $\hat{\beta}(r, \rho, \lambda)$ we construct equilibria with positive saving. The hyperbolic discounting case is an interesting special case, although our results cover more general situations.

The next result is a well known adaptation of a result by Laibson (1996) and applies in the special case quasi-hyperbolic case, without asset limits and assuming power utility.

**Theorem 1** (Linear Markov Equilibria). *Suppose preferences satisfy*

\[
\beta(c) = \hat{\beta} \leq 1 \quad \text{and} \quad U_1(c) = \frac{1}{1 - \sigma} c^{1 - \sigma}
\]
and that there is no asset limit, $a = 0$. Then there is a linear Markov equilibrium with $\hat{c}(a) = \psi a$. Moreover, there is strictly positive saving, i.e. $\psi < r$, if $\bar{\beta} > \hat{\beta}(r, \rho, \lambda)$ and dissaving, i.e. $\psi > r$, if $\bar{\beta} < \hat{\beta}(r, \rho, \lambda)$.

Proof. Appendix D.

Before presenting these results, we start with a very simple result for the borderline case where $\beta(c) = \hat{\beta}(r, \rho, \lambda)$, showing that an equilibrium exists where the agent does not save nor dissave.

**Theorem 2 (Zero Savings).** Assume that $\beta(c) = \hat{\beta}(r, \rho, \lambda)$ for all $c > ra$. Then, $(V, W) = (\bar{V}, \bar{W})$ and $\hat{c}(a) = ra$ is a Markov equilibrium.

Proof. Appendix E.

### 4.1 Dissaving

Our first result constructs an equilibrium with dissaving, when $\beta(c) < \hat{\beta}(r, \rho, \lambda)$. Since $\beta(c) < 1$ by Assumption 1, this condition is ensured if $r \leq \rho$. However, it also holds when $r > \rho$ if $\beta(c)$ is low enough.

**Theorem 3 (Dissaving).** Suppose that $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ for all $c \geq ra$. Then there exists a Markov equilibrium with $\hat{c}(a) = ra$ and $\hat{c}(a) \geq ra$ for $a \geq \bar{a}$; indeed, the latter inequality is strict, $\hat{c}(a) > ra$, except possibly on a countable set of steady state points where $\hat{c}(a) = ra$.

Proof. Appendix E.

Recall from Proposition 2 that when $r > \rho$ the commitment solution eventually involved positive savings, with consumption and assets growing without bound. Thus, dissaving here is due to the lack of commitment and a time-inconsistency problem spurred by the disagreement regarding utility functions. When the ratio of marginal utilities is low enough the time-inconsistency problem is sufficiently severe to induce dissaving. The condition that $\beta(c) < \hat{\beta}(r, \rho, \lambda)$ is more likely to hold for higher $\lambda$ and lower $r$, which is intuitive since more disagreement is required if the interest rate is higher or if the agent is likely to remain in power longer.

### 4.2 Saving

We now consider the opposite case where disagreement is low enough and show that $\beta(c) > \hat{\beta}(r, \rho, \lambda)$ provides conditions for positive savings to emerge in equilibrium. Since
our approach is constructive we must fully characterize saving for high enough asset levels. The following assumption is helpful in this regard.

**Assumption 2.** Either (a) there is no disagreement for high enough consumption levels: \( \beta(c) = 1 \) for all \( c \geq \bar{c} \) for some \( \bar{c} > 0 \); or (b) disagreement is constant and the utility functions are power functions for high enough consumption levels: \( \beta(c) = \bar{\beta} < 1 \) and \( U_1(c) = \frac{1}{1-\sigma}c^{1-\sigma} \) for all \( c \geq \bar{c} \) for some \( \bar{c} > 0 \).

We use Assumption 2 to provide an boundary condition at some high enough level for the asset. In case (a) we consider an equilibrium that coincides with commitment solution for high enough assets. In case (b) we consider the equilibrium with a constant savings rate in Theorem 1. Although we do have a borrowing constraint, with positive savings (because \( \bar{\beta} > \hat{\beta} \)) this constraint is never binding. Using either of these two boundary conditions, the next result constructs an equilibrium with positive savings by solving the HJB system.

**Theorem 4 (Saving).** Suppose that \( \beta(\bar{c}) > \hat{\beta}(r, \rho, \lambda) \) and that Assumption 1 and 2 hold. Then there exists \( \hat{a} < \frac{\bar{c}}{r} \) such that if \( a > \hat{a} \) there exists a Markov equilibrium with \( \hat{c}(a) < ra \) and \( \hat{c}'(a) > 0 \) for \( a \geq \hat{a} \).

**Proof.** Appendix F

### 4.3 A Numerical Example

The following example illustrates the previous two theorems.

**Example 1.** Let the utility for the agent in power be given by

\[
U_1(c) = \frac{c^{1-\sigma}}{1-\sigma}
\]

for \( \sigma > 0 \) and let disagreement be given by

\[
\beta(c) = \begin{cases} 
\bar{\beta} \left( \frac{\alpha (\frac{c}{\bar{c}})^{\gamma} + 1 - \alpha}{\hat{\beta}} \right) & \text{if } c \leq \bar{c}, \\
\bar{\beta} & \text{if } c \geq \bar{c}.
\end{cases}
\]

with \( \alpha, \bar{\beta} \leq 1 \) and \( \gamma > 0 \). Under this specification \( \beta(c) \) is an increasing and continuous function of \( c \), reaching a plateau of \( \bar{\beta} \) at \( c = \bar{c} \). The implied \( U_0 \), which defined by \( U_1 \) and

---

\[18\text{Another simple way to provide an upper boundary condition is to assume a maximum asset level } a. \text{ When } \beta(ra) > \hat{\beta}, \text{ we can construct an equilibrium with savings by using the boundary } \hat{c}(a) = ra, V(a) = \hat{V}(a) \text{ and } W(a) = \hat{W}(a). \text{ Proposition 4 effectively involves a similar construction, with positive savings to the left of an interior steady-state.}\]
\( r = 0.05 \)

\[ ra \bar{c} \hat{c}(a) \]

\[ Consumption \]

\( r = 0.06 \)

\[ ra \hat{c}(a) \]

Figure 1: Consumption Functions under Different Interest Rates

\( \beta(c) \) up to a constant, is concave as long as \( \gamma \alpha \leq \sigma \).\footnote{For \( c < \bar{c}, U''_0(c) = \hat{\beta} \left( -\sigma(1 - \alpha) + \alpha (\xi) \gamma (\gamma - \sigma) \right) c^{-\sigma-1} < 0 \) if \( \gamma \alpha \leq \sigma \).} For our numerical examples we use the following parameters: \( \rho = 0.05, \sigma = \frac{1}{5}, \bar{\beta} = \frac{3}{4}, \alpha = \frac{3}{4}, \gamma = \frac{\sigma}{\alpha}, \lambda = 0.05, \bar{c} = 5 \) and \( \bar{a} = 30 \).

Figure 1 depicts the policy function \( \hat{c}(a) \) against \( ra \). The left panel sets \( r = 0.05 \) which ensures that \( \beta(c) < \hat{\beta}(r, \rho, \lambda) \) so that Theorem 3 applies. Since \( \hat{c}(a) > ra \) the agent is dissaving and assets fall \( \dot{a} < 0 \). The consumption function is concave. Indeed, it has infinite slope at the lower asset limit \( a \) and this limit is reached in finite time.

The right panel sets \( r = 0.06 \) ensuring that \( \beta(c) > \hat{\beta}(r, \rho, \lambda) \) so that Theorem 4 applies. Since \( \hat{c}(a) < ra \) the agent is saving and assets rise \( \dot{a} > 0 \) without bound. For a high enough level of assets the consumption function becomes linear, coinciding with the linear equilibrium, provided in Theorem 1, for the quasi-hyperbolic model with \( \beta(c) = \bar{\beta} \) (lower dotted line). This provides the boundary condition needed in our construction. For lower levels of assets the equilibrium consumption function is nonlinear and slightly convex.

5 Non-Uniform Disagreement

We are now interested in situations where disagreement varies sufficiently, so that we do not have the conditions for global dissaving or saving equilibria.
5.1 Decreasing Disagreement: Poverty Traps

We start with the case where disagreement falls with consumption and lies on both sides of our $\hat{\beta}(r, \rho, \lambda)$. Given the fact that $\beta(c) < 1$ by Assumption 1, this requires $r > \rho$.

**Assumption 3.** The ratio $\beta(c)$ is weakly increasing in c and crosses $\hat{\beta}(r, \rho, \lambda)$.

Since $\beta(c) > \hat{\beta}(r, \rho, \lambda)$ for high consumption levels, Theorem 4 and provides an equilibrium with savings as long as $a \geq \hat{a}$. However, if $a < \hat{a}$ the construction fails and there is no equilibrium with positive saving for all asset levels. Indeed, since $\beta(r\hat{a}) < \hat{\beta}(r, \rho, \lambda)$ dissaving seems like a natural outcome for lower asset levels, following Theorem 3. This opens the door to poverty traps: positive saving above some cutoff asset level and dissaving below this cutoff. Intuitively, for lower asset levels disagreement is severe, lowering the incentive to save and perpetuating the time-inconsistency problem. At high enough asset levels disagreements are lower and the agent is able to save and overcome the time inconsistency problem. Indeed, reaching regions with lower disagreements and time inconsistency may provide an additional incentive to save.

The next result essentially combines the constructions underlying Theorems 3 and 4, using that of Theorem 3 for low asset levels and that of Theorem 4 for high assets. The asset cutoff must be set at a point where the agent in power is indifferent between following the saving and dissaving paths.

**Theorem 5 (Poverty Trap).** Suppose Assumptions 1, 2 and 3 hold. Then there exists a cutoff $a^*$ and a Markov equilibrium with saving for $a > a^*$ and weak dissaving for $a < a^*$, i.e. $\hat{c}(a) < ra$ for $a > a^*$ and $\hat{c}(a) \geq ra$ for $a < a^*$.

**Proof.** Appendix G

This result still leaves open the possibility that $a^* = a$, so that there is no region with dissaving, or that $a^* = \infty$, so that there is no region with positive saving. Indeed, these outcome may occur in some cases. Intuitively, dissaving may not occur if the difference between $\beta(c)$ and $\hat{\beta}(r, \rho, \lambda)$ are small. Likewise, positive savings may not occur if $\beta$ is too low at the top. Ensuring an interior $\hat{a}$ requires additional assumptions.

The next result provides sufficient conditions for $a^*$ to be interior.

**Proposition 3.** Suppose Assumptions 1, 2a and 3 and $a < \frac{\xi}{\rho}$. Then there exists $r^* > \rho$ such that for all $r \in (\rho, r^*)$ the cutoff $a^*$ defined in Theorem 5 is interior, i.e. $a < a^* < \infty$.

**Proof.** Appendix G.
Of course poverty traps with interior $a^*$ also arise away from these sufficient conditions, as the next example illustrates.\footnote{In an earlier version of this paper, we show the existence of Markov equilibrium with a poverty trap for $U_1(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $\beta(c) = \beta$ for $c < \bar{c}$ and $\beta(c) = 1$ for $c \geq \bar{c}$.}

**Example 2.** The following numerical example illustrates a poverty trap under the conditions of Theorem 5. We start with the parameters in Example 1. We set the borrowing constraint $a$ at 15. As depicted in Figure 2, $\hat{c}(a) \geq ra$ for $a < a^*$ and $\hat{c}(a) < ra$ for $a > a^*$. Thus, there is poverty trap for all $a < a^*$.

By construction $a^* > \hat{a}$. In this case, a lower borrowing constraint not only affects the equilibrium value functions and consumption function at lower asset levels, but can also affect the behavior above $\hat{a}$. In particular, at $\hat{a} < a < a^*$, the agent saves and grows when $a > \hat{a}$ (Theorem 4). However, the agent dissaves and shrinks its asset when $a < \hat{a}$. Therefore, decreasing the borrowing constraint can lead to worse dynamic equilibrium behavior. Bernheim et al. (2014) obtains a similar result using sub-game perfect equilibrium as a solution concept, instead of the Markov equilibrium used in this paper.

### 5.2 Increasing Disagreement: Convergence

We now consider the reverse situation, when disagreement rises with spending. When $\beta(c)$ is decreasing the time-inconsistency problem is aggravated at higher asset levels. Intuitively, this provides a force for dissaving at high asset levels and positive savings at
low asset levels. If these forces are strong enough they may generate convergence to a unique steady state level of assets. We now provide one such result.

In order to prove our global convergence result we also require a downward jump.\footnote{This requires $U_0$ to feature a concave kink. Kinks were ruled out by our prior assumption that $U_0$ is everywhere differentiable, but we relax this assumption for the next result.} We conjecture that the convergence may also obtain for continuous $\beta(c)$ if the function is sufficiently decreasing.\footnote{In this case $c^*$ is uniquely determined by $\beta(c^*) = \tilde{\beta}$.}

**Assumption 4.** Suppose $\beta(c)$ is decreasing and continuous everywhere except at $c^* > 0$ where

$$\lim_{c \uparrow c^*} \beta(c) > \lim_{c \downarrow c^*} \beta(c).$$

Under this assumption, we show that there is a Markov equilibrium defined over some interval $[a, \infty)$ with an interior steady state at $a^* \equiv \frac{c^*}{\lambda}$, for some $r$, that is stable. The agent saves positively when assets are below $a^*$ and dissaves (weakly) when assets are above $a^*$.

**Proposition 4.** Suppose Assumption 1 and 4 hold. Then there is an interval of interest rates $(r, \bar{r})$ with $\rho < r$ with the property that for any interest rate $r \in (r, \bar{r})$ there exists a Markov equilibrium over an interval $[a, \infty)$ containing a unique stable stationary state $a^* = \frac{c^*}{\lambda} > a$, i.e. $\hat{c}(a^*) = ra^*$, $\hat{c}(a) < ra$ for $a < a^*$ and $\hat{c}(a) \geq ra$ for $a > a^*$ and $\hat{c}(a) > ra$ strictly in some neighborhood to the right of $a^*$.

In addition, if $1 - \beta(c) > -\frac{U''(c)c}{U'(c)}$ for $c > c^*$ and $\lambda$ is sufficiently high then $\hat{c}(a) > ra$ for all $a > a^*$ and when $\beta(c)$ is constant for $c < c^*$ then we can guarantee $a = 0$.

**Proof.** Appendix H.\]

Intuitively, at high asset levels the time inconsistency problem leads to dissaving, while at low asset levels the fact that $r > \rho$ leads to positive savings. Thus, the variable time-inconsistency problem provides a force for convergence, despite a constant interest rate.

We now provide a local convergence result of a different nature. If we restrict asset choices to a local neighborhood, we just need $\beta$ to lie in an intermediate range to obtain local convergence. The following theorem formalizes this result.

**Theorem 6 (Local Indeterminacy).** Suppose $r > \rho$ and

$$\beta(r\tilde{a}) \in \left( \hat{\beta}, 1 - \frac{r - \rho}{\lambda} \right) \quad (12)$$
for some $\tilde{a} > 0$. There exists a Markov equilibrium over an interval $[a, \tilde{a}]$ containing $\tilde{a}$ as the unique stable stationary state i.e. $\hat{c}(\tilde{a}) = r\tilde{a}$, $\hat{c}(a) < ra$ for $a < \tilde{a}$ and $\hat{c}(a) > ra$ for $a > \tilde{a}$.

Proof. Appendix H.

This result is the continuous time version of the indeterminacy result emphasized in Krusell and Smith (2003). For any $\tilde{a}$ such that (12) holds, there is a Markov equilibrium with $\tilde{a}$ as the unique steady-state. In this sense there is local indeterminacy. However, to obtain the Markov equilibrium, we need to restrict assets choices to $[a, \tilde{a}]$ in some neighborhood of $\tilde{a}$. We have verified numerically that the equilibrium value and policy functions cannot be extended indefinitely the right. Indeed, $V(a) < \bar{V}(a)$ for all $a \in (\tilde{a}, \bar{a})$, so we cannot apply the construction procedure in Theorem 3, which required $V$ to cross $\bar{V}$ at some $a > \tilde{a}$.

6 Further Characterization: Discontinuous Equilibria, Multiplicity of Equilibria, and Instantaneous Gratification

Here we characterize equilibria further.

6.1 Continuous and Discontinuous Dissaving Equilibria

Theorem 3 shows the existence of Markov equilibria with dissaving. However, it still leaves open two possibilities. The equilibrium may be continuous, with a single steady state at the lower limit $a$ and strict dissaving above $\tilde{a}$. Alternatively, the Markov equilibrium may be discontinuous with multiple steady states, including $a$.

The following theorems sharpen the characterization of the Markov equilibria. Define the local curvature of the utility function

$$\sigma(U_1, c) \equiv -\frac{U_1''(c)c}{U_1'(c)}.$$ 

Because $U_1$ is strictly increasing and strictly concave, $\sigma(U_1, c) > 0$ for all $c > 0$.

The follow theorem presents sufficient conditions that guarantees the existence of continuous Markov equilibria.

23Consequently, this local indeterminacy result is not a multiple equilibrium result because the games restricted to different intervals $[a, \bar{a}]$ are different games. See Section 6.2 for a multiple equilibrium result in which different equilibria are defined in the same asset space $[\bar{a}, \infty)$.
Theorem 7. When \( r < \rho \), there exists a continuous Markov equilibrium with dissaving, i.e. \( V, W, \hat{c} \) continuously differentiable and \( \hat{c}(a) > ra \) for \( a > \tilde{a} \) and \( \hat{c}(\tilde{a}) = ra \). Moreover, the consumption function is strictly increasing with \( \hat{c}'(a) > 0 \).

Proof. Appendix I. \qed

Theorem 8. Assume that there exists \( \epsilon > 0 \) such that, for all \( c > ra \),

\[
\frac{1 - \sigma(U_1,c)}{\beta(c)} > 1 + \epsilon,
\]

(13)

There exists \( \bar{r} > \rho \) such that for any \( r \in [\rho, \bar{r}] \), a continuous Markov equilibrium exists with dissaving, i.e. \( V, W, \hat{c} \) are continuously differentiable and \( \hat{c}(a) > ra \) for all \( a > \tilde{a} \) and \( \hat{c}(\tilde{a}) = ra \). Moreover, \( \hat{c}'(a) > 0 \). Moreover, \( \bar{r} \) is strictly increasing in \( \lambda \) and \( \lim_{\lambda \to \infty} \bar{r}(\lambda) = \infty \).

Proof. Appendix I. \qed

Theorem 8 requires that, in order to have continuous Markov equilibria, risk-aversion coefficient (or more precisely, the intertemporal elasticity of substitution) is not too high relative to the degree of time-inconsistency, i.e. risk-aversion is not strong enough to prevent too much dissaving implied by time-inconsistency. As we shown in Subsection 7.1 below, under power functions, condition (13) is the opposite of the condition in Harris and Laibson (2013) that guarantees the existence of equilibrium when \( \lambda = \infty \).24 Numerically, we find that under condition (13), Markov equilibria exist for any finite \( \lambda \) but the sequence of value functions in Markov equilibria when \( \lambda \to \infty \) does not converge to a solution of the limiting system (15). In addition, the sequence of policy functions diverges, i.e., \( \hat{c}(a) \) goes to \( \infty \) as \( \lambda \) goes to infinity for any \( a > \tilde{a} \). Example 3 below presents a case in which a continuous Markov equilibrium exists.

Theorems 7 and 8 stand in contrast with the result in Chatterjee and Eyigungor (2014). The authors show that, in discrete time, when \( r \leq \rho \) and \( U_1 \) and \( U_0 \) are power utility functions (quasi-hyperbolic discounting case), a continuous Markov equilibrium does not exist. In continuous time Theorems 7 and 8 show that continuous Markov equilibria exists.25

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24 When \( U_1(c) = \frac{c^{1-\sigma}}{1-\sigma} \) and \( U_0(c) = \beta U_1(c) \), Harris and Laibson (2013) require that \( 1 - \beta < \sigma \).

25 One important observation that enables Chatterjee and Eyigungor (2014) to rule out continuous Markov equilibria is that in any Markov equilibrium there must be an interval of asset near the debt limit (the flat spot), such that starting from any asset level in this interval the decision maker chooses to go to the debt limit (a discontinuity in the policy function must appear close to the end of this interval). This observation holds even if the switching of power is stochastic as in this paper, as opposed to being deterministic (after one period) as in their paper. However, this observation does not apply in continuous time because starting from any asset level strictly above the debt limit, it takes a non-negligible amount of time to reach
The follow theorem presents sufficient conditions that guarantee the existence of discontinuous Markov equilibria.

**Theorem 9.** Assume \( r > \rho \) and that either \( U_1 \) is bounded above, or there exists \( \bar{c} \) and \( \sigma \) such that for all \( c > \bar{c} \),

\[
\sigma(U_1, c) > \sigma.
\]  
(14)

Let \((V_\lambda, W_\lambda, \hat{c}_\lambda)\) denote a Markov equilibrium with dissaving constructed in Theorem 3 under Poisson switching rate \( \lambda \). There exists \( \lambda_1 \) such that for all \( \lambda \geq \lambda_1 \),

\[
a_1(\lambda) = \inf \{ a > \underline{a} \mid V_\lambda(a) = \underline{V}(a) \} < \infty,
\]

and \( \lim_{\lambda \to \infty} a_1(\lambda) = \underline{a} \).

**Proof.** The details of the proofs are in Appendix I.

This theorem implies that for \( \lambda > \lambda_1 \), there exists a discontinuous Markov equilibrium because starting from \( a(\lambda) \), we can apply Theorem 3 to construct an equilibrium with \( a(\lambda) \) stands in for \( a \). By combining this equilibrium with the earlier equilibrium with \( V_\lambda, W_\lambda, \hat{c}_\lambda \) defined over \( [\underline{a}, a_1(\lambda)] \), we obtain a Markov equilibrium over \( [\underline{a}, \infty) \), in which \( W \) and \( \hat{c} \) are discontinuous at \( \hat{a} \). Although, \( V \) is continuous. The proof of this theorem also implies that when \( \lambda \to \infty \) (the instantaneous gratification limit in Harris and Laibson (2013)), the equilibrium value functions \( V_\lambda \) and \( W_\lambda \), which jumps each time \( V_\lambda \) crosses \( \underline{V} \), both converge to \( \underline{W}(a) \).

Example 3 below presents a discontinuous Markov equilibrium with many steady-states.

### 6.2 Multiplicity

When \( r > \rho \) but close to \( \rho \) Theorem 9 and 8 turn out to be compatible. This implies that there exist multiple Markov equilibria. The following theorem formalizes this result.

**Theorem 10.** Assume \( r > \rho \) and conditions (13) and (14) are satisfied. There exists \( \bar{\lambda} \), such that for each \( \lambda > \bar{\lambda} \), there exist at least two distinct Markov equilibria.

**Proof.** Appendix I

The following numerical example illustrates the result.

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26However, \( \bar{W} \) is not a solution to the limiting system (15).
**Example 3.** We use the standard following functional form for quasi-hyperbolic discounting:

\[
U_1(c) = \frac{c^{1-\sigma}}{1-\sigma} \\
U_0(c) = \bar{\beta} U_1(c),
\]

with the following parameters, similar to the ones from Harris and Laibson (2013),

\[
\rho = 0.05 \quad r = 0.05 \quad \sigma = \frac{3}{4}.
\]

When \( \bar{\beta} \) is sufficiently low, so that \( \sigma < 1 - \beta \) and \( \lambda \) is sufficiently high Theorems 9 and 8 can be applied.\(^{27}\) Figure 3 describes two equilibria when \( \beta = \frac{1}{5} \) and \( \lambda = 12 \). The equilibrium on the left panels (the upper panel depicts value functions and the lower panel depicts the policy function), has continuous value and policy functions as constructed in Theorem 8. Given that \( \lambda \) is sufficiently high, another equilibrium with multiple steady states exists and is depicted in the right panels.

The issue of multiple Markov equilibria and discontinuous Markov equilibria is also discussed in Dutta and Sundaram (1993) in a class of two-agent resource game.

In Subsection 7.1 below, we show that when there are multiple equilibria and a continuous equilibrium exists, a noisy perturbation to the game (either through random return to saving, or through random income) will select the continuous equilibrium.

### 6.3 Instantaneous Gratification Limit

When \( \lambda \to \infty \) we have \( V(a) \to W(a) \) and the HJB ODE system (8) becomes

\[
\tilde{c}(a) = \arg\max_c U_1(c) + W'(a)(ra - c), \\
\rho W(a) = U_0(\tilde{c}(a)) + W'(a)(ra - \tilde{c}(a)). \tag{15}
\]

Following Harris and Laibson (2013), we define a Markov equilibrium in the instantaneous gratification (IG) limit, \( \lambda \to \infty \), as a solution, \( W \), to the limiting HJB equation (15) and that \( W \) is continuous and piecewise differentiable, displaying only convex kinks. In the next two propositions, we show that there are two cases in which \( W \) exists and is continuously differentiable.

\(^{27}\) \( \lambda = 12 \) corresponds to monthly switching frequency.
Figure 3: Multiple Equilibria under Quasi-Hyperbolic Discounting.
The following proposition corresponds to the IG limit of the saving equilibrium in Theorem 4.

**Proposition 5.** Assume Assumption 2 and \( \beta(\bar{c}) > \hat{\beta}(r, \rho, \lambda = \infty) = \frac{\rho}{r} \). There exists \( \hat{a} < \frac{\bar{c}}{r} \), such that if \( a > \hat{a} \), a continuous Markov equilibrium at \( \lambda = \infty \) (instantaneous gratification limit) exists with the property that \( \hat{c}(a) < ra \) and \( \hat{c}'(a) > 0 \) for \( a \geq a \). Moreover, under Assumption 2a and \( \beta(c) \) is weakly increasing in \( c \), there exists \( r_1 > \rho \) such that for all \( r \in (\rho, r_1] \), \( \hat{a} \) is strictly positive.

**Proof.** Appendix J.

The following proposition characterizes the IG limit when \( r < \rho \).

**Proposition 6.** Assume that \( r < \rho \), and \( \frac{1 - \beta(c)}{\sigma(U_1, c)} \) is weakly decreasing in \( c \) and there exist \( \epsilon, \bar{\sigma} > 0 \) such that

\[ \epsilon < \frac{1 - \beta(c)}{\sigma(U_1, c)} < 1 - \epsilon \]

and \( \sigma(U_1, c) \leq \bar{\sigma} \) for all \( c \geq ra \). Moreover there exists \( \bar{c} > \frac{ra}{1 - \frac{1 - \beta(ra)}{\sigma(U_1, ra)}} \) such that

\[ \frac{U_0(\epsilon) - U_0(ra)}{\bar{c} - ra} = U'_1(\bar{c}). \]

Then a continuous IG limit exists.

**Proof.** Appendix J. In addition, we show that the last condition is satisfied if \( \sigma(U_1, c) \equiv \bar{\sigma} \) and \( \beta(c) \equiv \bar{\beta} \). In this case we also provide a closed form for the IG limit.

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**7 Extensions: Uncertainty and General Equilibrium**

In this section we present two extensions, one with uncertainty and the other one with general equilibrium.

**7.1 Uncertainty**

We investigate two ways of introducing uncertainty to our basic framework. In the first formulation, the rate of return on saving is stochastic as in Harris and Laibson (2013). In the second formulation, income is stochastic.
7.1.1 Return Uncertainty

Following Harris and Laibson (2013), we assume that the rate of return on saving is stochastic, in particular

$$da_t = ra_t dt + va_t dB_t$$

(16)

where $B_t$ is the standard Brownian motion. Because of uncertainty, the change of variables in Section 2 no longer applies, so we work directly with strictly positive income $y > 0$ and a borrowing limit $a_t \geq 0$.

Following the discussion in Section 2, a Markov equilibrium can be characterized by

$$\rho V(a) = \max_{c \geq 0} \{ U_1(c) + V'(a)(ra + y - c) + \lambda (W(a) - V(a)) + \frac{1}{2} \nu^2 a^2 V''(a) \},$$

(17a)

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra + y - \hat{c}(a)) + \frac{1}{2} \nu^2 a^2 W''(a),$$

(17b)

where $\hat{c}(a)$ denotes the solution to the maximization in (17a), which includes the additional constraint $c \leq y$ when $a = 0$.

At the limit $\lambda \to \infty$ (instantaneous gratification), this system becomes

$$\hat{c}(a) = \arg \max_{c \geq 0} \{ U_1(c) + W'(a)(ra + y - c) \},$$

(18a)

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(ra + y - \hat{c}(a)) + \frac{1}{2} \nu^2 a^2 W''(a),$$

(18b)

with the additional constraint $c \leq y$ when $a = 0$.

**Theorem 11.** Assume that

$$F(c) \equiv \beta(c) - 1 + \sigma(U_1, c) > 0$$

(19)

for all $c > 0$. Moreover, $F(c)$ is weakly increasing and $\frac{U_1'(c)}{U_1''(c)}$ is strictly increasing in $c$. System (18) admits a solution that is bounded above by the value function of a time-consistent consumer with utility $U_0$.

**Proof.** Appendix K. The proof follows closely Harris and Laibson (2013). Indeed, we show that there exists a time-consistent consumer with wealth dependent utility function $\hat{U}(c,a)$, such that the HJB equation for the value function of this consumer, $\hat{V}$, is equivalent to the equation (18) for $W$. Thus $\hat{V}$ is a solution to (18). Uniqueness also holds but the proof would require a new theory of viscosity solution with the Hamiltonian having a discontinuity at the boundary.\(^{28}\) Indeed, from the construction in Appendix K, $\hat{U}(c,a)$

\(^{28}\)Private communication with Pierre-Louis Lions and Benjamin Moll.
is continuous everywhere except for a discontinuity at \( a = 0 \). The discontinuity arises because of the borrowing constraint.

This result also guarantees that when the noise \( \nu \) goes to zero, solution \( W \) of (18) converges to a well-defined limit which is a solution of (15), or the value function of a deterministic IG-consumer.

To illustrate this result, we use the numerical example from Harris and Laibson (2013).

**Example 4.** The parameters are for the case of quasi-hyperbolic discounting in Harris and Laibson (2013) except for \( r = 0.07 \) and \( \lambda = \infty \) (Instantaneous Gratification Limit). Figure 4 shows the policy function, \( \hat{c}(a) \) for the IG consumer when the standard deviation of noise changes, \( \nu = 0.17, 0.10, 0.07 \) and at the limit \( \nu = 0 \).

When \( \lambda < \infty \), however the limit when \( \nu \to 0 \) might not be well-defined. Figure 5 shows the policy function when \( \lambda = 10 \) for two different values of \( \nu = 0.17 \) and 0.10. This figure shows that the consumption function displays divergent behavior when \( \nu \) decreases (in contrast to the IG case).

**Example 5.** The parameters are for the case of time-dependent inconsistency in Example 1 except for the uncertainty in the rate of return on asset, equation (16). Figure 6 shows the policy function, \( \hat{c}(a) \) for the IG consumer for two values of the standard deviation.

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29In the Online Appendix, we provide a closed-form solution to the consumption function under this joint limit \( \lambda = \infty \) and \( \nu = 0 \).
of noise, \( \nu = 0.17 \) and \( \nu = 0.10 \). When \( \nu = 0.10 \), we observe that saving is positive for high values of \( a \) while negative for low values of \( a \). This (stochastic) behavior resembles a poverty trap described in Theorem 5.

### 7.1.2 Income Uncertainty

When the rate of return on saving is constant but the income process is stochastic,

\[
dy_t = \mu(y_t)dt + \nu^2(y_t)dB_t
\]

where \( B_t \) is the standard Brownian motion.

Following the discussion in Section 2, a Markov equilibrium can be characterized by

\[
\rho V (a, y) = \max_{c \geq 0} \left\{ U_1 (c) + \frac{\partial V}{\partial a} (a, y) (ra + y - c) + \lambda (W (a) - V (a)) \right\} + , \tag{20a}
\]

\[
\frac{\partial V (a, y)}{\partial y} \mu(y) + \frac{1}{2} \nu^2(y) \frac{\partial^2 V}{\partial y^2} (a, y)
\]

\[
\rho W (a, y) = U_0 (\hat{c} (a)) + \frac{\partial W}{\partial a} (a, y) (ra + y - \hat{c} (a)) + , \tag{20b}
\]

\[
\frac{\partial W (a, y)}{\partial y} \mu(y) + \frac{1}{2} \nu^2(y) \frac{\partial^2 W}{\partial y^2} (a, y)
\]

where \( \hat{c} (a) \) denotes the solution to the maximization in (20a), which includes the addi-
Figure 6: Consumption Functions under Wealth-Dependent Time-Inconsistency and Instantaneous Gratification

When \( \lambda = \infty \), Markov equilibrium can be characterized by

\[
\hat{c}(a, y) = \arg\max_{c \geq 0} \left\{ U_1(c) + \frac{\partial W}{\partial a}(ra + y - c) \right\},
\]

(21a)

\[
\rho W(a, y) = U_0(\hat{c}(a)) + \frac{\partial W}{\partial a}(ra + y - \hat{c}(a)) + \frac{\partial W}{\partial y} \mu(y) + \frac{1}{2} \nu^2(y) \frac{\partial^2 W}{\partial y^2},
\]

(21b)

which includes the additional constraint \( c \leq y \) when \( a = 0 \).

**Theorem 12.** Under the conditions in Theorem 11, the system (21) admits a solution which is bounded above by the value function of a time-consistent consumer with utility \( U_0 \).

**Proof.** Appendix K. Similar to the proof of Theorem 11, we show that \( W(a, y) \) exists as the value function of a time consistent \( \hat{U} \)-consumer. However, with income uncertainty, \( \hat{U} = \hat{U}(c, a, y) \), depends on \( c, a, y \) instead of \( c, a \) only as in Theorem 11.

In the numerical example below, we use the Ornstein-Uhlenbeck process for income shocks:

\[
dy_t = -\theta y_t \log(y_t) dt + \sigma_y y_t dB_t.
\]

(22)

\[\text{Equivalently,}\]

\[
d \log y_t = -\theta \log(y_t) dt + \sigma_y dB_t.
\]

(23)

\[\text{Moll (2014) uses this process to model persistent productivity shocks and study the effect of financial frictions on aggregate productivity.}\]
We look at the limit $\sigma_y \to 0$. For the parameters in Example 3, there are multiple equilibria and one of them is continuous. We find that as noise goes to zero, the solution converge to the continuous equilibrium.

**Example 6.** The parameters are in Example 3, except that income is uncertain and follows the process (22). We vary the standard deviation of income shocks, $\sigma_y = 1, 0.5, 0.1$. The left panel of Figure 7 shows the consumption function, for the median value of income, as well as the continuous consumption function without uncertain income in Example 3. We observe that as noise decreases, consumption increases (and saving decreases) due to the precautionary motive. As noise goes to zero, the consumption functions converge to the continuous consumption function without uncertainty. As a comparison, we also plot consumption functions (right panel) when return on saving is uncertain and follows the process (16). In this case, as noise decreases, saving becomes less risky, so saving increases and consumption decreases. However, as noise goes to zero, the consumption functions also converge to the continuous consumption function without uncertainty.

### 7.2 General Equilibrium

In this subsection, we extend our analysis to the case with closed economy in which the return on asset is determined by a concave production function $f(a)$ such that $f'(a) > 0$.
and $f''(a) < 0$, as in Krusell and Smith (2003). The marginal rate of return on asset, $f'(a)$, is endogenously determined by (which corresponds to a general equilibrium microfoundation) and is strictly decreasing in the total supply of asset $a$, instead of being exogenously given at $r$ in the previous analysis.

The equation describing the evolution of asset (3) now becomes

$$\dot{a}_t = f(a_t) - c_t.$$ Combining this evolution with the behavior of the decision maker and the value functions $V, W$ of the agents in the economy, the dynamics of the economy is characterized by

$$\rho V(a) = \max_{c \geq 0} \{U_1(c) + V'(a)(f(a) - c) + \lambda(W(a) - V(a))\}, \quad (23a)$$

$$\rho W(a) = U_0(\hat{c}(a)) + W'(a)(f(a) - \hat{c}(a)). \quad (23b)$$

As for constant returns, we define $(\bar{V}, \bar{W})$ as the pair of value functions which corresponds to the case in which consumption is set so that asset stays constant, $c(a) = f(a)$:

$$\bar{V}(a) = \frac{1}{\rho + \lambda} U_1(f(a)) + \frac{\lambda}{(\rho + \lambda)\rho} U_0(f(a))$$

$$\bar{W}(a) = \frac{1}{\rho} U_0(f(a)). \quad (24)$$

When there is no time-inconsistency, i.e., $\beta(c) = 1$ for all $c > 0$, the standard analysis of the neoclassical growth model in continuous time shows us that the economy admits a unique steady state $a^{ss}$, determined by

$$f'(a^{ss}) = \rho. \quad (25)$$

Going back to our model, we assume that at (and beyond) the consumption level implied by this steady state level of asset, $c^{ss} = f(a^{ss})$, there is no time inconsistency, i.e. $\beta(f(a^{ss})) = 1$. It is easy to verify that $a^{ss}$ is also a steady state of (23). Below $a^{ss}$, there are two forces that induce the decision maker to save and reach $a^{ss}$ in finite time: 1) the standard neoclassical force coming from high marginal rate of return on asset when asset is low and 2) the saving out of time inconsistency force analyzed in Subsection 4.2, which requires that $\beta(c)$ to be increasing.

Formally, under the following assumption, we show that starting from an asset level

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32 Barro (1999) and Krusell et al. (2002) study a competitive version of this model in which agents take the marginal rate of return on capital as given.
below $s^s$ the decision maker will saving until asset reaches $a^s$.

**Assumption 5.** Assume Assumption 2a and at $a^s$ determined by (25), $f(a^s) > c$.

Under this assumption, we can prove a result similar to Theorem 4 for exogenous return.

**Theorem 13.** Assume Assumptions 1 and 5. There exists $\hat{a} \in [0, a^s)$ such that for any $a \geq \hat{a}$, a Markov equilibrium exists with $\hat{c}(a) < f(a)$ and $\hat{c}'(a) > 0$ for $\hat{a} \leq a \leq a^s$.

**Proof.** Appendix L

When $a < \hat{a}$, we can extend the construction in Theorem 5 to this case, if

$$\left(\frac{1}{\rho + \lambda} + \frac{\lambda}{(\rho + \lambda) \rho} \beta(f(a))\right) f'(a) < 1,$$

for all $\hat{a} \leq a \leq \hat{a}$.

## 8 Conclusion

Our analysis shows that the time-inconsistency problem is very tractable in a continuous time setup, providing insights into the hyperbolic discounting model. This has also allowed us to consider extensions to situations where the time inconsistency problem is variable. We have seen that interesting dynamics may emerge in these cases.
References


Appendix

A General Properties

A.1 Proof of Proposition 1

We show existence and uniqueness at the same time. By the Envelope Theorem

\[ U'_1(c) = h'(\hat{x}(c)) = g'(\hat{z}(c)). \]

Moreover,

\[ U'_0(c) = h'(\hat{x}(c))\hat{x}'(c). \]

Therefore, \( \hat{x}'(c) = \frac{U'_0(c)}{U'_1(c)} \). Thus

\[ \hat{x}(c) = \int_0^c \frac{U'_0(\tilde{c})}{U'_1(\tilde{c})} d\tilde{c}. \]

and

\[ \hat{z}(c) = c - \hat{x}(c) = \int_0^c \left( 1 - \frac{U'_0(\tilde{c})}{U'_1(\tilde{c})} \right) d\tilde{c}, \]

is increasing in \( c \) because \( 1 - \frac{U'_0(\tilde{c})}{U'_1(\tilde{c})} \geq 0 \). So \( h(x) \) and \( g(z) \) are uniquely determined (up to constants) by

\[ h'(\hat{x}(c)) = U'_1(c) \]

and

\[ g'(\hat{z}(c)) = U'_1(c). \]

\( h \) and \( g \) are increasing because \( h', g' > 0 \) and are concave because \( \hat{x} \) and \( \hat{z} \) are increasing in \( c \) and \( U'_1 \) is decreasing in \( c \).

A.2 Roots of Hamilton-Jacobi-Bellman Equations

Write the HJB equation as

\[ (\rho + \lambda) V(a) - \lambda W(a) = H (V'(a), a), \]  \( (26) \)

where

\[ H (p, a) \equiv \sup_c \{ U_1(c) + p (ra - c) \}. \]  \( (27) \)

The next lemma characterizes the function \( H \).
Lemma 2. For any a, the function $H(\cdot, a)$ defined by (27) is continuous, strictly convex and continuously differentiable for $p > 0$; has a unique interior minimum at $p = U'(ra)$; satisfies $\lim_{p \to \infty} H(p, a) = \infty$ and $H(0, a) = \lim_{p \to 0} H(p, a) = U_1(\infty)$.

Proof. For any $p > 0$ a maximum is attained on the right hand side of (27) uniquely by the first-order condition $U'_1(c) = p$. This implies that $H(a, \cdot)$ differentiable with derivative $H_p(p, a) = ra - (U'_1)^{-1}(p)$. This derivative is continuous and strictly increasing. Thus, $H(p, a)$ is strictly convex in $p$. Since $H_p(U'_1(ra), a) = 0$ then $p = U'_1(ra)$ is the unique minimum. Since $H(a, \cdot)$ is strictly convex it follows that $\lim_{p \to \infty} H(p, a) = \infty$.

Finally, by definition $H(0, a) \equiv \sup_c U_1(c) = \lim_{c \to \infty} U_1(c)$. This also coincides with $\lim_{p \to 0} H(p, a)$ since

$$\lim_{p \to 0} H(p, a) = \lim_{c \to \infty} (U_1(c) + U'_1(c)(ra - c)) \leq \lim_{c \to \infty} U_1(c),$$

$$\lim_{p \to 0} H(p, a) \geq \lim_{p \to 0} (U_1(p^{-\frac{1}{2}}) + p(ra - p^{-\frac{1}{2}})) = \lim_{c \to \infty} U_1(c).$$

This has immediate implications for the possible solutions to equation (26).

Lemma 3. Consider solutions $p = V'(a)$ to equation (26), if

Case 1. $(\rho + \lambda) V(a) - \lambda W(a) < U_1(ra)$, then no solution exists;

Case 2. $(\rho + \lambda) V(a) - \lambda W(a) = U_1(ra)$, then the unique solution is given by $p = U'_1(ra)$;

Case 3. $U_1(ra) < (\rho + \lambda) V(a) - \lambda W(a) \leq U_1(\infty)$, then exactly two solutions $p_1$ and $p_2$ exist and $0 \leq p_1 < U'_1(ra) < p_2$;

Case 4. $U_1(\infty) < (\rho + \lambda) V(a) - \lambda W(a)$, then a unique solution exists and $U'_1(ra) < p$.

Given Lemma 3, we define the following subsets of $\mathbb{R}^3$:

$$E \equiv \{(a, V, W) | a > 0 \text{ and } (\rho + \lambda) V - \lambda W > U_1(ra)\},$$

$$E_0 \equiv \{(a, V, W) | a > 0 \text{ and } U_1(\infty) > (\rho + \lambda) V - \lambda W > U_1(ra)\},$$

$$E_s \equiv \{(a, V, W) | a > 0 \text{ and } (\rho + \lambda) V - \lambda W = U_1(ra)\}.$$

Lastly $E = E \cup E_s$, and $E_0 = E_0 \cup E_s$. Notice that $E_s$ corresponds to the set of singular points of the HJB equation (8) as an implicit ODE.

Using Lemma 3 we now rewrite system (8) as explicit ODEs. There are two systems to consider, depending on whether we consider the high or lower root.

Definition 1. Let $R_l(a, V, W)$ denote the lower root $p = V'(a)$ of equation (26). By Lemma 3, $R_l$ is well-defined over $E_0$ and is continuous in $a, V, W$. Let $S_l(a, V, W)$ denote the
associated solution to $W'$ in equation (8b), so that

$$S_l(a, V, W) = \frac{U_0(\hat{c}(a)) - \rho W}{\hat{c}(a) - ra}$$

with $\hat{c}(a) = (U_1')^{-1}(V'(a)) = (U_1')^{-1}(R_l(a, V, W)) > ra$, defined over $E_0$. By the Implicit Function Theorem, $R_l$ and $S_l$ are continuously differentiable in $a, V, W$ over $E_0$.

Using $R_l$ and $S_l$, system (8) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_l(a, V, W) \\ S_l(a, V, W) \end{pmatrix}.$$ \hspace{1cm} (28)

This ODE is regular around $(a, V, W) \in E_0$. Around any regular point we can apply standard extension results (for example, Picard–Lindelöf theorem or Cauchy–Lipschitz theorem; see Hartman (2002) for a comprehensive exposition) to show that, the ODE (28) admits a unique solution $(V(a), W(a)) = (v, w)$.

The next definition is analogous, but using the higher root of equation (26).

**Definition 2.** Let $R_h(a, V, W)$ be the higher root for $p = V'(a)$ of equation (26). By Lemma 3, $R_h$ is well-defined over $E$ and is continuous in $a, V, W$. Let $S_h(a, V, W)$ be the associated value $W'$ in equation (8b), so that

$$S_h(a, V, W) = \frac{\rho W - U_0(\hat{c}(a))}{ra - \hat{c}(a)},$$

where $\hat{c}(a) = (U_1')^{-1}(V'(a)) = (U_1')^{-1}(R_h(a, V, W)) < ra$, defined over $E$. By the Implicit Function Theorem, $R_h$ and $S_h$ are continuously differentiable in $(a, V, W)$ over $E$.

Using $R_h$ and $S_h$, system (8) can be represented as an explicit ODE

$$\begin{pmatrix} V'(a) \\ W'(a) \end{pmatrix} = \begin{pmatrix} R_h(a, V, W) \\ S_h(a, V, W) \end{pmatrix}.$$ \hspace{1cm} (29)

This ODE is regular around any $(a, V, W) \in E$. Just as with (28), standard extension results apply whenever $(a, V, W)$ is regular.

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For any solution $x(a)$ to an ODE $x'(a) = F(x(a))$. If $F$ is continuously differentiable then $x$ is twice continuously differentiable, and $x''(a) = \nabla F(x) \cdot x' = \nabla F(x) \cdot F(x)$.\footnote{For any solution $x(a)$ to an ODE $x'(a) = F(x(a))$. If $F$ is continuously differentiable then $x$ is twice continuously differentiable, and $x''(a) = \nabla F(x) \cdot x' = \nabla F(x) \cdot F(x)$.}
A.3 Full Commitment Solution

A.3.1 Proof of Proposition 2

From the evolution of asset (3),
\[ a_t = \int_0^\infty e^{-rs_c} dt s \geq a. \] (30)

Let \( \bar{U}(c, t) = e^{-\lambda t} U_1(c) + (1 - e^{-\lambda t}) U_0(c) \). The objective function of the commitment problem can be rewritten as \( \int_0^\infty e^{-\rho \bar{U}(c_t) dt} \).

Consider a variation where we decrease \( c_t \) by \( \varepsilon > 0 \) and increase \( c_{t+s} \) by \( e^{rs} \varepsilon \) then this increases the objective locally if and only if
\[ \bar{U}_c(c_{t+s}, t + s) < e^{-(\rho - r)s} \bar{U}_c(c_t, t). \]

This variation satisfies the budget constraint and weakly increases asset at any time so it is feasible. Thus a necessary condition for an optimum is that
\[ \bar{U}_c(c_{t+s}, t + s) \leq e^{(\rho - r)s} \bar{U}_c(c_t, t). \]

In particular
\[ \bar{U}_c(c_t, t) \leq e^{(\rho - r)t} \bar{U}_c(c_0, 0) \to 0, \]
as \( t \to \infty \).

Moreover, by Assumption 1,
\[ \bar{U}_c(c_t, t) = e^{-\lambda t} U'_1(c_t) + (1 - e^{-\lambda t}) U'_0(c_t) \geq \beta U'_1(c_t). \] (31)

Therefore \( U'_1(c_t) \to 0 \) as \( t \to \infty \). By the INADA condition, this implies \( c_t \to \infty \) which then requires \( a_t \to \infty \), from (30).

Thus, there is a date \( T \) for which all \( t \geq T \) we have the borrowing constraint, \( a_t \geq a \), not binding. For any dates \( t \geq T \), we can perform the same variation as above but with \( \varepsilon < 0 \), thus we must have
\[ \bar{U}_c(c_{t+s}, t + s) = e^{(\rho - r)s} \bar{U}_c(c_t, t). \]
at an optimum. This then implies that \( c_t \) is monotone for all \( t \geq \hat{T} \) for some \( \hat{T} > T \).

Indeed, differentiating both sides with respect to \( s \), together with letting \( \mu = \bar{U}_c(c_T, T) > 0 \)
0 and using \( t \) standing for \( T + s \), we obtain
\[
\left( e^{-\lambda t} U''_1(c_i) + (1 - e^{-\lambda t}) U'_0(c_i) \right) \dot{c}_t = - (r - \rho) \mu e^{-(r-\rho) t} + \lambda e^{-\lambda t} (U'_1(c_i) - U'_0(c_i)) \\
\leq - (r - \rho) \mu e^{-(r-\rho) t} + \lambda e^{-\lambda t} U'_1(c_i)(1 - \beta) \\
\leq - (r - \rho) \mu e^{-(r-\rho) t} + \lambda e^{-\lambda t} \mu e^{-(r-\rho)(t-T)} \frac{1 - \beta}{\beta},
\]
where the first inequality comes from Assumption 1 that \( U'_0(c_i) > \beta U'_0(c_i) \) and the second inequality comes from (31). There exists \( \hat{T} > T \) such that \( (r - \rho) \mu > \lambda e^{-\lambda t} \mu e^{(r-\rho)T} \frac{1 - \beta}{\beta} \) for all \( t \geq \hat{T} \). Moreover, \( e^{-\lambda t} U''_1(c_i) + (1 - e^{-\lambda t}) U'_0(c_i) < 0 \), therefore \( \dot{c}_t > 0 \) for all \( t \geq \hat{T} \).

From (30), \( \dot{a}_t > 0 \) for all \( t \geq \hat{T} \).

### A.3.2 Properties of the Commitment Solution

Suppose Assumption 2a, i.e. there exists \( \bar{c} \) such that \( U'_1(c) = U'_0(c) \) for all \( c \geq \bar{c} \). In this case
\[
U_1(c) = U_0(c) + \bar{u}
\]
for all \( c \geq \bar{c} \) for some \( \bar{u} \in \mathbb{R} \). Without loss of generality in what follows we normalize \( \bar{u} = 0 \).

Because \( \beta(c) \leq 1 \), from Assumption 1, \( U_1(c) \leq U_0(c) \) for all \( c \leq \bar{c} \). It is then immediate that
\[
V_{sp}(a_0) \leq V_0^*(a_0) \equiv \max \int e^{-\rho t} U_0(c(t)) \quad \text{s.t.} \quad a_0 = \int_0^\infty e^{-rt} c_i dt.
\]

Consider the variations as in the proof of Proposition 2, for an optimum to the maximization problem in the right-hand side,
\[
U'_0(c_i) = e^{(\rho-r)t} U'_0(c_0)
\]
which gives \( c_t \) as a strictly increasing function of \( c_0 \) and is strictly increasing in \( t \) because \( \rho < r \). We then pin down \( c_0 \) as a function of \( a_0 \) from
\[
a_0 = \int_0^\infty e^{-rt} c_t dt.
\]
The right-hand side is strictly increasing in \( c_0 \), so \( c_0 = \dot{c}_0(a_0) \) is uniquely pinned down from this equation and is strictly increasing in \( a_0 \).

Let \( a_{\mu} = \max \left\{ \int_0^\infty e^{-rt} (U'_0)^{-1} e^{(\rho-r)t} U'_0(c) \ dt, a \right\} \). Then, for \( a_0 \geq a_{\mu} \), \( \dot{c}_0(a_0) \geq \bar{c} \).
Because \( c_t \) is strictly increasing in \( t \), \( c(t) \geq \bar{c} \) for all \( t \geq 0 \), and \( a(t) = \int^\infty_0 e^{-rt} c_{t+s} ds \geq a_0 \geq \bar{a} \) for all \( t \geq 0 \). Then it follows that \( V_{sp}(a_0) = V_0^*(a_0) \) for all \( a_0 \geq a_u \).

It is standard to show that the value function \( V_0^*(a) \) is concave and differentiable and satisfies an HJB equation,

\[
\rho V_0^*(a) = \max_c U_0(c) + V_0^*(a)(ra - c),
\]

and the policy function \( \hat{c}(a) = \arg \max_c U_0(c) + V_0^*(a)(ra - c) \) gives the optimal solution \( \tilde{c}_0(a_0) \) defined above. Thus \( \hat{c} \) satisfies \( \hat{c}(a_0) \geq \bar{c} \) for all \( a_0 \geq a_u \). It then follows that system (8) holds for \( V = V_0^* \) and \( W = V_0^* \) for all \( a_0 \geq a_u \).

### A.4 Useful Observations

The following general properties of the solutions to the system (8) is also important for their characterization.

**Lemma 4.** Assume that \( V, W \) and \( \hat{c} \) constitutes a solution to the system (8). If \( V \) and \( W \) are continuously differentiable and \( V \) is twice differentiable at \( a \), then

\[
(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a) (ra - \hat{c}(a)) \tag{32}
\]

and if \( \hat{c}(a) \neq ra \):

\[
\hat{c}'(a) = \frac{1}{U'_1(\hat{c}(a)))} \frac{(\rho + \lambda - r) V'(a) - \lambda W'(a)}{ra - \hat{c}(a)} \tag{33}
\]

**Proof.** Differentiating (8a) with respect to \( a \), we obtain

\[
(\rho + \lambda) V'(a) - \lambda W'(a) = U'_1(\hat{c}(a)))\hat{c}'(a) + V'(a)(r - \hat{c}'(a)) + V''(a) (ra - \hat{c}(a)) .
\]

Combining this with (11) and rearranging yield (32). Now differentiating (11) with respect to \( a \),

\[
U''_1(\hat{c}(a)))\hat{c}'(a) = V''(a),
\]

or equivalently \( \hat{c}'(a) = \frac{V''(a)}{U''_1(\hat{c}(a))} \), which together with (32) yields (33). \( \square \)

Lastly, we will also use the follow result to connect the comparison between \( \beta(.) \) and \( \hat{\beta} \) to the comparison between the slope of \( V \) and \( U'_1 \).

**Lemma 5.** For \( a > 0 \), \( \beta(ra) < \hat{\beta}(r, \rho, \lambda) \) if and only if

\[
V'(a) < U'_1(ra). \tag{34}
\]
And \( \beta(ra) = \hat{\beta}(r, \rho, \lambda) \) if and only if \( \overline{V}'(a) = U'_1(ra) \).

**Proof.** Using the definition (10) for \( \overline{V} \), we have

\[
\overline{V}'(a) = \frac{r}{\rho + \lambda} U'_1(ra) + \frac{\lambda r}{(\rho + \lambda) \rho} U'_0(ra).
\]

The condition that \( \hat{\beta}(ra) < \hat{\beta}(r, \rho, \lambda) \) is equivalent to

\[
\frac{r}{\rho + \lambda} U'_1(ra) + \frac{\lambda r}{(\rho + \lambda) \rho} U'_0(ra) < U'_1(ra).
\]

The result then follows. Likewise, for the case with \( \beta(ra) = \hat{\beta}(r, \rho, \lambda) \). \( \square \)

**B A Single-Crossing Property**

The following simple result on the comparison between two functions plays a crucial role in the characterization of the solutions to the system (8). Although this result is very simple, we do not know of any reference, so include it here for completeness.

**Lemma 6.** Let \( f \) and \( g \) be two continuously differentiable functions defined over \([a, \bar{a}]\). Consider the subset satisfying the requirements that (1) \( f(a) \geq g(a) \); and (2) if \( f(a) = g(a) \) for some \( a \in [a, \bar{a}] \) then \( f'(a) > g'(a) \). Then \( f(a) > g(a) \) for all \( a \in [a, \bar{a}] \).

**Proof.** First, observe that, if \( f(a) = g(a) \), by property 2, \( f'(a) > g'(a) \), therefore \( f(a) > g(a) \) in the right neighborhood of \( a \). If \( f(a) > g(a) \), we obtain the same result by continuity. Now, we prove the lemma by contradiction. Assume that, there exists \( \tilde{a} \in [a, \bar{a}] \) such that \( f(\tilde{a}) \leq g(\tilde{a}) \). By the Intermediate Value Theorem, we can assume that \( f(\tilde{a}) = g(\tilde{a}) \), without loss of generality. Now let \( a^* = \inf \{a \in [a, \bar{a}] : f(a) = g(a) \} \). By continuity \( f(a^*) = g(a^*) \). Moreover, \( a^* > a \) because \( f(a) > g(a) \) in the right neighborhood of \( a \). By property 2, \( f'(a^*) > g'(a^*) \). Together with \( f(a^*) = g(a^*) \), this implies, \( f(a) < g(a) \) in the left neighborhood of \( a^* \). Therefore by the Intermediate Value Theorem, there exists \( a^{**} \in (a, a^*) \) such that \( f(a^{**}) = g(a^{**}) \). This contradicts the definition of \( a^* \) which is the infimum. \( \square \)

We also use a few variations of this lemma.

**Variation 1.** if 1) \( f(\bar{a}) \geq g(\bar{a}) \), and 2) if \( f(a) = g(a) \), for some \( a < \bar{a} \), then \( f'(a) < g'(a) \), we have \( f(a) > g(a) \) for all \( a \in [a, \bar{a}] \).

**Variation 2.** We can also relax condition 2, by the condition that if \( f(a) = g(a) \) then \( f'(\bar{a}) > g'(\bar{a}) \) in the left neighborhood of \( a \). Indeed, in the proof above, if \( f(a^*) = g(a^*) \)
and $f'(\bar{a}) > g'(\bar{a})$ in the left neighborhood of $a^*$, then for $a$ in the left neighborhood of $a^*$,

$$f(a) = f(a^*) - \int_a^{a^*} f'(\bar{a}) d\bar{a}$$

$$= g(a^*) - \int_a^{a^*} f'(\bar{a}) d\bar{a}$$

$$< g(a^*) - \int_a^{a^*} g'(\bar{a}) d\bar{a} = g(a).$$

We can then proceed as in the remaining of the proof. This variation is useful when $f'$ or $g'$ are not well-defined at some $a$.

## C Proofs for Lemma 1

### C.1 Proof of Lemma 1

For any $\epsilon > 0$ sufficiently small, indeed satisfying

$$\epsilon < \lim_{\epsilon \to +\infty} \frac{U_1(c) - U_1(\epsilon)}{\lambda},$$

consider the solution $(V_\epsilon, W_\epsilon)$ to the ODE (28) satisfying the initial condition

$$(V_\epsilon(a_0), W_\epsilon(a_0)) = (V(a_0), W(a_0) - \epsilon). \quad (35)$$

Given that (28) is regular around $a_0$, we can apply standard ODE existence results to show that $(V_\epsilon, W_\epsilon)$ exists and is unique over some interval $[a_0, a_0 + \omega]$ that depends on $\epsilon$. We will use $(V_\epsilon, W_\epsilon)$ to construct the equilibrium described in Lemma 1 as follows:

First, Lemma 7 shows that there exists an $\omega > 0$ and $\bar{\epsilon} > 0$ such that for $0 < \epsilon < \bar{\epsilon}$ such that $(V_\epsilon, W_\epsilon)$ are defined over $[a_0, a_0 + \omega]$, independent of $\epsilon$. Second, Lemma 9 shows that for $0 < \epsilon < \bar{\epsilon}$, the slopes of $V_\epsilon$ and $W_\epsilon$ are uniformly bounded over $[a_0, a_0 + \omega]$. Finally, using 1) and 2) and applying the Dominated Convergence Theorem, we show that $(V_\epsilon, W_\epsilon)$ converges to $(V, W)$ for a subsequence $\epsilon_N \to 0$ and $(V, W)$ is a solution to system (8).

We now describe this last step in detail. Lemma 7 shows that there exist $\omega > 0$ and $\bar{\epsilon} > 0$ such that: for any $\epsilon < \bar{\epsilon}$ the solution $(V_\epsilon(a), W_\epsilon(a))$ are defined over $[a_0, a_0 + \omega]$ and
that $V_e(a) > \overline{V}(a)$ for all $a \in (a_0, a_0 + \omega]$.Lemma 9 implies that for all $a \in [a_0, a_0 + \omega]$, 
\[
0 \leq W'_e(a) \leq U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra),
\]
\[
0 \leq V'_e(a) \leq U'_1(ra).
\]

We now construct a solution $(V, W)$ of the original system (8) over $[a_0, a_0 + \omega]$ as follows. Because the derivatives $V'_e$ and $W'_e$ are uniformly bounded, the families of functions \{$V_e$\} and \{$W_e$\} defined over $[a_0, a_0 + \omega]$ are uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, there exists a sequence $\epsilon_N$ such that $(V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ converges uniformly to continuous functions $(V, W)$. We now show that this candidate $(V, W)$ is a solution to (8).

For any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$ we have
\[
V_{\epsilon_N}(a_1) - V_{\epsilon_N}(a_2) = \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) \, da.
\]

Since $R_l$ is continuous
\[
\lim_{N \to \infty} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = R_l(a, V(a), W(a)).
\]

Moreover, by Lemma 9, $R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a))$ is uniformly bounded over $[a_1, a_2]$: $0 \leq R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) = V'_{\epsilon_N}(a) \leq U'_1(ra) < U'_1(ra_0)$. Therefore, by the Dominated Convergence Theorem,
\[
\lim_{N \to \infty} \int_{a_1}^{a_2} R_l(a, V_{\epsilon_N}(a), W_{\epsilon_N}(a)) \, da = \int_{a_1}^{a_2} R_l(a, V(a), W(a)) \, da.
\]
Thus,

$$V(a_1) - V(a_2) = \lim_{N \to \infty} (V_{\epsilon N}(a_1) - V_{\epsilon N}(a_2))$$

$$= \lim_{N \to \infty} \int_{a_1}^{a_2} R_l(a, V_{\epsilon N}(a), W_{\epsilon N}(a)) \, da$$

$$= \int_{a_1}^{a_2} R_l(a, V(a), W(a)) \, da.$$ \hfill (36)

Similarly, for any two points $a_1 < a_2$ in the interval $[a_0, a_0 + \omega]$ we have

$$W_{\epsilon N}(a_1) - W_{\epsilon N}(a_2) = \int_{a_1}^{a_2} S_l(a, V_{\epsilon N}(a), W_{\epsilon N}(a)) \, da.$$

By choosing $\omega$ sufficiently small, the last property in Lemma 7 applies for each $a \in (a_0, a_0 + \omega]$. We show that, $(a, V(a), W(a)) \in E_0$ for each $a \in (a_0, a_0 + \omega]$ and

$$\lim_{N \to \infty} S_l(a, V_{\epsilon N}(a), W_{\epsilon N}(a)) = S_l(a, V(a), W(a)).$$ \hfill (37)

Indeed, from the definition of $V_{\epsilon N}$, $W_{\epsilon N}$, $(\rho + \lambda)V_{\epsilon N}(a) - \lambda W_{\epsilon N}(a) > U_1(ra)$. Therefore, by pointwise convergence, $(\rho + \lambda)V(a) - \lambda W(a) \geq U_1(ra)$. We show by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. Assume to the contrary that $(\rho + \lambda)V(a) - \lambda W(a) = U_1(ra)$. From the last property of Lemma 7, $V_{\epsilon N}(a) \geq \bar{V}(a) + \gamma_a$ for $\epsilon_N < \epsilon_a$. Therefore, by pointwise convergence, $V(a) \geq \bar{V}(a) + \gamma_a$. This, together with the contradiction assumption, implies that

$$W(a) < \bar{W}(a) - \frac{\lambda}{\rho + \lambda} \gamma_a.$$

In addition, by the continuity of $R_l$ and by pointwise convergence,

$$\hat{c}_{\epsilon N}(a) = (U_1')^{-1}(R_l(a, V_{\epsilon N}(a), W_{\epsilon N}(a))) \to ra$$
as $N \to \infty$. Consequently, there exists $\delta \in (0, 1)$ such that for $N$ sufficiently high,

$$S_l(a, V_{eN}(a), W_{eN}(a)) = \frac{\rho W_{eN}(a) - U_0(\hat{c}_{eN}(a))}{ra - \hat{c}_{eN}(a)} \left( \frac{\rho \bar{W}(a) \hat{c}_{eN}(a) - U_0(ra)}{ra - \hat{c}_{eN}(a)} \right) = +\infty,$$

as $N \to \infty$, which contradicts the boundedness of $S_l(a, V_{eN}(a), W_{eN}(a))$ shown in Lemma 9. Therefore, we have shown by contradiction that $(\rho + \lambda)V(a) - \lambda W(a) > U_1(ra)$. By the continuity of $S_l$ in $E_0$, we obtain the limit (37).

Since $0 < S_l(a, V_{eN}(a), W_{eN}(a)) < U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra) < U'_0(ra_1) + \frac{\rho}{\lambda} U'_1(ra_1)$, by the Dominated Convergence Theorem, we can take the limit and conclude that

$$W(a_1) - W(a_2) = \int_{a_1}^{a_2} S_l(a, V(a), W(a)) \, da. \tag{38}$$

In addition, $R_l, S_l$ are continuous over $E_0$, therefore (36) and (38) imply that $(V, W)$ is a solution to ODE (28) over $(a_0, a_0 + \omega_0]$: this immediately implies that (8) holds for all $a \in (a_0, a_0 + \omega_0]$.

Next we show that (8) holds at $a = a_0$. We showed that $V'(a_0) = R_l(a_0, V(a_0), W(a_0))$, so equation (8a) holds at $a = a_0$. Since $(V(a_0), W(a_0)) = \lim_{N \to \infty} (V_{eN}(a_0), W_{eN}(a_0)) = (\bar{V}(a_0), \bar{W}(a_0))$ this implies that $V'(a_0) = U'_1(ra_0)$. Since $V'(a_0) = U'_1(ra_0)$, this gives $\hat{c}(a_0) = ra_0$, and so equation (8b) holds.

Having established the existence of $(V, W)$, we turn to showing Properties 1) and 2).

Property 1: Notice that the right derivative of $V$ at $a_0$, $V'(a_0) = R_l(a_0, V(a_0), W(a_0)) = U'_1(ra_0) > \bar{V}'(a_0)$, by Lemma 5. Together with $V(a_0) = \bar{V}(a_0)$, we have $V(a) > \bar{V}(a)$ in a neighborhood to the right of $a_0$. Restricting $\omega$ so that $a_0 + \omega$ lies in this neighborhood, we obtain the first property in Lemma 1.

Property 2: Because $\hat{c}(a) = (U'_1)^{-1}(V'(a))$ and $\lim_{a \downarrow a_0} V'(a) = \lim_{a \uparrow a_0} R_l(a, V(a), W(a)) = U'_1(ra_0), \lim_{a \downarrow a_0} \hat{c}(a) = ra_0$.

By (33) in Lemma 4,

$$\hat{c}'(a) = \frac{1}{U'_1(\hat{c}(a))} \frac{(\rho + \lambda - r) V'(a) - \lambda W'(a)}{ra - \hat{c}(a)}.$$

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From the derivation (41) in Lemma 9,
\[ W'(a) = \lim_{\epsilon \to 0} W'_\epsilon(a) \leq \lim_{\epsilon \to 0} \left( U_0(ra) + \frac{\rho}{\lambda} U_1'(ra) - \frac{\rho}{\lambda} V'_\epsilon(a) \right), \]
and \( V'(a) = \lim_{\epsilon \to 0} V'_\epsilon(a) \). Therefore
\[ \epsilon'(a) \geq \frac{1}{U_1'(\epsilon'(a))} \lim_{\epsilon \to 0} \frac{((2\rho + \lambda - r)V'_\epsilon(a) - \lambda U_0(ra) - \rho U_1'(ra))}{ra - \epsilon(a)}. \]
Because \( \epsilon(a) \to ra_0 \) as \( a \to a_0 \), \( \lim_{a \to a_0} (ra - \epsilon(a)) = 0 \). Moreover,
\[ \lim_{(a,\epsilon) \to (a_0,0)} ((2\rho + \lambda - r)V'_\epsilon(a) - \lambda U_0(ra) - \rho U_1'(ra)) = (\rho + \lambda - r) U_1'(ra_0) - \lambda U_0'(ra_0) > 0, \]
where the last inequality comes from the fact that
\[ \frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) < \frac{\rho}{r} \left( \frac{\lambda + \rho - r}{\lambda} \right) \leq \frac{\lambda + \rho - r}{\lambda}, \]
if \( r \geq \rho \) and
\[ \frac{U_0'(ra_0)}{U_1'(ra_0)} = \beta(ra_0) \leq 1 < \frac{\rho + \lambda - r}{\lambda}, \]
if \( r < \rho \). As a result, \( \lim_{a \to a_0} \epsilon'(a) = +\infty \). We have established the second property in Lemma 1.

C.2 Supporting Results for Lemma 1

The proof of Lemma 1 given above draws on the following results.

The first lemma below shows that there exists \( \omega > 0 \) and \( \bar{\epsilon} \) such that for each \( \epsilon \in (0, \bar{\epsilon}) \), the solution \((V_\epsilon, W_\epsilon)\) to ODE (28) are defined over \([a_0, a_0 + \omega]\) and \( V_\epsilon(a) > \overline{V}(a) \). The proof of this lemma uses Lemma 8 that follow.

**Lemma 7.** There exist \( \omega > 0 \) and \( \bar{\epsilon} > 0 \) such that for every \( \epsilon \in (0, \bar{\epsilon}) \), \((V_\epsilon(a), W_\epsilon(a))\) constructed in the proof of Lemma 1 is defined on \([a_0, a_0 + \omega]\). Moreover, \( V_\epsilon(a) > \overline{V}(a) \) for all \( a \in (a_0, a_0 + \omega] \). Lastly, there exists \( \omega_0 < \omega \) such that for each \( a \in (a_0, a_0 + \omega_0] \), there exist \( \epsilon_a, \gamma_a > 0 \) such that \( V_\epsilon(a) > \overline{V}(a) + \gamma_a \) for all \( 0 < \epsilon < \epsilon_a \).

**Proof.** Let \( \bar{\epsilon}_1 = \frac{1}{\lambda} (U_1(\infty) - U_1(ra_0)) > 0 \). For \( 0 < \epsilon < \bar{\epsilon}_1 \), let \([a_0, \bar{a}_\epsilon]\) denote the (right)
maximal interval of existence for \((V_e, W_e)\).\(^{34}\) Lemma 8 shows that if \(\bar{a}_e < \infty\) then
\[
(\bar{a}_e, V_e(\bar{a}_e), W_e(\bar{a}_e)) \in E_s.
\]
In addition, \(V_e(\bar{a}_e) \leq \bar{V}(\bar{a}_e)\).
Because \(R_l\) is continuous,
\[
\lim_{\epsilon \to 0} R_l(a_0, V_e(a_0), W_e(a_0)) = R_l(a_0, \bar{V}(a_0), \bar{W}(a_0)) = U'_{l}(ra_0) > \bar{V}'(a_0).
\]
Therefore, there exists \(\bar{\epsilon}_2 > 0\), such that \(V'_e(a_0) = R_l(a_0, V_e(a_0), W_e(a_0)) > \bar{V}'(a_0)\) for \(0 < \epsilon < \bar{\epsilon}_2\). In this case, \(V_e(a) > \bar{V}(a)\) in some neighborhood to the right of \(a_0\).
For \(0 < \epsilon < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)\), let
\[
\bar{a}_\epsilon = \sup \{a \in (a_0, \bar{a}_e) : V_e(a') > \bar{V}(a') \text{ for all } a' \in (a_0, a)\}.
\]
Because \(V_e(a) > \bar{V}(a)\) in some neighborhood to the right of \(a_0\), as shown above, \(\bar{a}_\epsilon > a_0\).
We show by contradiction that there exist \(\omega > 0\) and \(0 < \tilde{\epsilon} < \min(\bar{\epsilon}_1, \bar{\epsilon}_2)\), such that \(\bar{a}_\epsilon > a_0 + \omega\) for all \(\epsilon < \tilde{\epsilon}\). Assume that this is not true, then there exists a sequence \(\epsilon_N \to 0\) such that \(\lim_{N \to \infty} \bar{a}_{\epsilon_N} = a_0\).
Because \(V_{\epsilon_N}\) is continuous, \(V_{\epsilon_N}(\bar{a}_{\epsilon_N}) \geq \bar{V}(\bar{a}_{\epsilon_N})\) (otherwise, \(V_{\epsilon_N}(a) < \bar{V}(a)\) in the some neighborhood to the left of \(\bar{a}_{\epsilon_N}\), which contradicts the definition of \(\bar{a}_e\)). If \(V_{\epsilon_N}(\bar{a}_{\epsilon_N}) > \bar{V}(\bar{a}_{\epsilon_N})\), then \(\bar{a}_{\epsilon_N} < \bar{a}_{\epsilon_N}\), because if \(\bar{a}_{\epsilon_N} < \infty\) then \(V_{\epsilon_N}(\bar{a}_{\epsilon_N}) \leq \bar{V}(\bar{a}_{\epsilon_N})\) as shown in Lemma 8. This also contradicts the definition of \(\bar{a}_{\epsilon_N}\), because \(V_{\epsilon_N}(a)\) is defined and is strictly greater than \(\bar{V}(a)\) in a neighborhood of \(\bar{a}_{\epsilon_N}\). Therefore \(V_{\epsilon_N}(\bar{a}_{\epsilon_N}) = \bar{V}(\bar{a}_{\epsilon_N})\).
By the Mean Value Theorem, there exists \(a^*_{\epsilon_N} \in [a_0, \bar{a}_{\epsilon_N}]\) such that
\[
\frac{V_{\epsilon_N}(\bar{a}_{\epsilon_N}) - V_{\epsilon_N}(a_0)}{\bar{a}_{\epsilon_N} - a_0} = V'_{\epsilon_N}(a^*_{\epsilon_N}) = \frac{\bar{V}(\bar{a}_{\epsilon_N}) - \bar{V}(a_0)}{\bar{a}_{\epsilon_N} - a_0} \tag{39}
\]
and by the definition of \(V_e, W_e\):
\[
V'_{\epsilon_N}(a^*_{\epsilon_N}) = R_l(a^*_{\epsilon_N}, V_{\epsilon_N}(a^*_{\epsilon_N}), W_{\epsilon_N}(a^*_{\epsilon_N}))
\]
By the monotonicity of \(V_e\) and \(W_e\) shown in Lemma 9,
\[
\bar{V}(a_0) < V_{\epsilon_N}(a^*_{\epsilon_N}) < \bar{V}(\bar{a}_{\epsilon_N})
\]
\(^{34}\)The definition of the maximal interval of existence is standard in the ODE literature. See, for example, Hartman (2002).
and
\[ W_{\epsilon N}(a_0) = \overline{W}(a_0) - \epsilon_N < W_{\epsilon N}(a^*_N). \]

Moreover, from the upper bound on \( W'_\epsilon \) shown in Lemma 9 (using \( V_{\epsilon N}(a) \geq \overline{V}(a) \) for \( a \in (a_0, \bar{a}_{\epsilon N}) \)):
\[
W_{\epsilon N}(a^*_N) \leq W_{\epsilon N}(a_0) + \left( U'_0(ra_0) + \frac{\rho}{\lambda} U'_1(ra_0) \right) (a^*_N - a_0).
\]

Besides, by the contradiction assumption, \( \lim_{N \to \infty} a^*_N = \lim_{N \to \infty} \bar{a}_{\epsilon N} = a_0 \). Therefore, by the Squeeze Principle, using the four inequalities above, we obtain
\[
\lim_{N \to \infty} V_{\epsilon N}(a^*_N) = \overline{V}(a_0)
\]
\[
\lim_{N \to \infty} W_{\epsilon N}(a^*_N) = \overline{W}(a_0).
\]

Thus, together with the continuity of \( R_I \) and (39), we obtain
\[
\lim_{N \to \infty} R_I(a^*_N, V_{\epsilon}(a^*_N), W_{\epsilon}(a^*_N)) = R_I(a_0, \overline{V}(a_0), \overline{W}(a_0)) = U'_1(ra_0)
\]
\[
= \lim_{N \to \infty} \frac{\overline{V}(\bar{a}_{\epsilon N}) - \overline{V}(a_0)}{\bar{a}_{\epsilon N} - a_0} = \overline{V}'(a_0).
\]

This leads to the desired contradiction because Lemma 5 for \( a = a_0 \) implies that \( \overline{V}'(a_0) < U'_1(ra_0) \).

Finally, we show the last property by contradiction. Assume that it does not hold. Then there exists a sequence \( a_N \to a_0 \) such that for each \( N \), there exists a sequence \( \epsilon_{N,M} \to 0 \) such that \( V_{\epsilon_{N,M}}(a_N) \to \overline{V}(a_N) \). By choosing \( M \) sufficiently high, we have \( 0 < \epsilon_{N,M} < \frac{1}{N} \) and
\[
\left| \frac{V_{\epsilon_{N,M}}(a_N) - \overline{V}(a_N)}{a_N - a_0} \right| < \frac{1}{N}.
\]

By the Mean Value Theorem, there exists \( \bar{a}_N \in [a_0, a_N] \) such that
\[
\frac{V_{\epsilon_{N,M}}(a_N) - \overline{V}(a_N)}{a_N - a_0} = \frac{V_{\epsilon_{N,M}}(a_N) - V_{\epsilon_{N,M}}(a_0) + V_{\epsilon_{N,M}}(a_0) - \overline{V}(a_N)}{a_N - a_0}
\]
\[
= V'_{\epsilon_{N,M}}(\bar{a}_N,M) - \overline{V}'(\bar{a}_{N,M}).
\]

Therefore
\[
\left| V'_{\epsilon_{N,M}}(\bar{a}_N,M) - \overline{V}'(\bar{a}_{N,M}) \right| < \frac{1}{N}. \tag{40}
\]
However,
\[ V_{\epsilon,N,M}'(\hat{a}_{N,M}) = R_l(\hat{a}_{N,M}, V_{\epsilon,N,M}(\hat{a}_{N,M}), W_{\epsilon,N,M}(\hat{a}_{N,M})) , \]
and by Lemma 9, as \( N, M \to \infty \) \( V_{\epsilon,N,M}(\hat{a}_{N,M}) \to \overline{V}(a_0) \) and \( W_{\epsilon,N,M}(\hat{a}_{N,M}) \to \overline{W}(a_0) \). Therefore by the continuity of \( R_l \),
\[ V_{\epsilon,N,M}'(\hat{a}_{N,M}) \to R_l(a_0, \overline{V}(a_0), \overline{W}(a_0)) = U_1'(ra_0). \]
Because \( \hat{a}_{N,M} \to a_0 \)
\[ \nabla'(\hat{a}_{N,M}) \to \nabla'(a_0). \]
Combining the last two limits with (40), we have \( U_1'(ra_0) = \nabla'(a_0) \), which contradicts condition (34) for \( a = a_0 \) that \( U_1'(ra_0) > \nabla'(a_0) \). Therefore by contradiction, the last property holds.

**Lemma 8.** Consider the (right) maximal interval of existence, \([a_0, \hat{a}]\) for the solution \((V_\epsilon, W_\epsilon)\) to the ODE (28) with the initial condition (35) and \( 0 \leq \epsilon < \frac{1}{\lambda}(U_1(\infty) - U_1(ra_0)) \). If \( \hat{a} < \infty \), then \( \lim_{a \uparrow \hat{a}} V_\epsilon(a) = V(\hat{a}) \) and \( \lim_{a \uparrow \hat{a}} W_\epsilon(a) = W(\hat{a}) \) and \((\hat{a}, V_\epsilon(\hat{a}), W_\epsilon(\hat{a})) \in E_s \). In addition, \( V_\epsilon(\hat{a}) \leq \nabla'(\hat{a}) \).

**Proof.** By Lemma 9, \( V_\epsilon'(a), W_\epsilon'(a) > 0 \). Therefore, the limits \( \lim_{a \uparrow \hat{a}} V_\epsilon(a) = V_\epsilon(\hat{a}) \) and \( \lim_{a \uparrow \hat{a}} W_\epsilon(a) = W_\epsilon(\hat{a}) \) exist. In addition, since \( V_\epsilon'(a) = U_1'(\hat{c}_\epsilon(a)) < U_1'(ra), V_\epsilon(\hat{a}) < \infty \) and \( W_\epsilon(\hat{a}) \leq \frac{(\rho + \lambda)V_\epsilon(\hat{a}) - U_1'(ra)}{\lambda} < \infty \). By Hartman (2002, Theorem 3.1), \((\hat{a}, V_\epsilon(\hat{a}), W_\epsilon(\hat{a})) \) must lie in the boundary of \( E_0 \), i.e. Case 1: \((\rho + \lambda)V_\epsilon(\hat{a}) - \lambda W_\epsilon(\hat{a}) = U_1(\infty) \) or Case 2: \((\rho + \lambda)V_\epsilon(\hat{a}) - \lambda W_\epsilon(\hat{a}) = U_1(ra) \). We first rule out Case 1 by showing that \((\rho + \lambda)V_\epsilon(\hat{a}) - \lambda W_\epsilon(\hat{a}) < U_1(\infty) \).

If \( U_1(\infty) = \infty \), this is obvious. Now if \( U_1(\infty) < \infty \). Let \( a(t) \) satisfies \( a(0) = \hat{a} \) and \( \frac{da(t)}{dt} = ra(t) - \hat{c}_\epsilon(a(t)) \) where \( \hat{c}_\epsilon(a) = (U_1')^{-1}(R_l(a, V_\epsilon(a), W_\epsilon(a))) > ra \). Consider the derivative:
\[
\frac{d}{dt} \left( e^{-(\rho + \lambda)t}V_\epsilon(a(t)) \right) = e^{-(\rho + \lambda)t} \left( -(\rho + \lambda)V_\epsilon(a(t)) + V_\epsilon'(a(t))(ra(t) - \hat{c}_\epsilon(a(t))) \right)
= e^{-(\rho + \lambda)t} \left( U_1(c(t)) + \lambda W_\epsilon(a(t)) \right),
\]
where the second equality comes from the fact that \( V_\epsilon \) is the solution of ODE (28). Let \( T \) denote the time at which \( a(t) \) reaches \( a_0 \) (\( T \) can be \(+\infty\)), then
\[
V_\epsilon(\hat{a}) = \int_0^T e^{-(\rho + \lambda)t}(U_1(\hat{c}_\epsilon(a(t))) + \lambda W_\epsilon(a(t)))dt + e^{-(\rho + \lambda)T}\nabla(a_0).
\]
Notice that, by Lemma 9, \( W_e(a) \) is strictly increasing in \( a \) and \( a_t \) is strictly decreasing in \( t \) because \( \ell_e(a(t)) > ra(t), W_e(a(t)) < W_e(a(0)) = W_e(\bar{a}) \). This implies

\[
V_e(\bar{a}) < \int_0^T e^{-(\rho + \lambda)t} (U_1(\ell_e(a(t))) + \lambda W_e(a(t))) dt + e^{-(\rho + \lambda)T} \nabla(a_0)
\]

\[
= \int_0^T e^{-(\rho + \lambda)t} (U_1(\infty) + \lambda W_e(\bar{a})) dt + e^{-(\rho + \lambda)T} \nabla(a_0)
\]

\[
= (1 - e^{-(\rho + \lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho + \lambda)T}) W_e(\bar{a}) + e^{-(\rho + \lambda)T} \nabla(a_0).
\]

By the definition of \( \nabla(a) \),

\[
\nabla(a_0) = \frac{1}{\rho + \lambda} U_1(ra_0) + \frac{\lambda}{\rho + \lambda} \tilde{W}(a_0) = \frac{1}{\rho + \lambda} U_1(ra_0) + \frac{\lambda}{\rho + \lambda} (W_e(a_0) + \epsilon)
\]

\[
< \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_e(a_0),
\]

since \( \epsilon < \frac{1}{\lambda} (U_1(\infty) - U_1(ra_0)) \). Thus

\[
V_e(\bar{a}) < (1 - e^{-(\rho + \lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho + \lambda)T}) W_e(\bar{a}) + e^{-(\rho + \lambda)T} \nabla(a_0)
\]

\[
< (1 - e^{-(\rho + \lambda)T}) \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} (1 - e^{-(\rho + \lambda)T}) W_e(\bar{a})
\]

\[
+ e^{-(\rho + \lambda)T} \left( \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_e(\bar{a}) \right) = \frac{1}{\rho + \lambda} U_1(\infty) + \frac{\lambda}{\rho + \lambda} W_e(\bar{a}).
\]

Therefore \( (\rho + \lambda) V_e(\bar{a}) < U_1(\infty) + \lambda W_e(\bar{a}) \) which is equivalent to the desired inequality.

So we must be in Case 2, i.e. \((\bar{a}, V(\bar{a}), W(\bar{a})) \in E_s\).

We show by contradiction that \( V_e(\bar{a}) < \nabla(\bar{a}) \). Assume to the contrary that \( V_e(\bar{a}) > \nabla(\bar{a}) \).

Then \( W_e(\bar{a}) > \tilde{W}(\bar{a}) \) because \(||\rho + \lambda \tilde{V}(\bar{a}) - \lambda \tilde{W}(\bar{a})\| = U_1(ra)\).

Since \( R_i \) is continuous over \( E_0, \lim_{a_t \uparrow \bar{a}} V_e'(a) = U_1'(ra) \) and \( \lim_{a_t \uparrow \bar{a}} \ell_e(a) = ra. \) Therefore

\[
\lim_{a_t \uparrow \bar{a}} W_e'(a) = \lim_{a_t \uparrow \bar{a}} \frac{U_0(\ell_e(a)) - \rho W_e(a)}{\ell_e(a) - ra} = \frac{U_0(ra) - \rho W_e(\bar{a})}{ra - ra} = -\infty,
\]

which contradicts the property that \( W_e' > 0 \) established in Lemma 9. So by contradiction, \( W_e(\bar{a}) \leq \tilde{W}(\bar{a}) \), and \( V_e(\bar{a}) \leq \nabla(\bar{a}) \).

The following lemma establishes the bounds on the derivative of \( V_e \) and \( W_e \) that are important to apply the Dominated Convergence Theorem in Lemma 1. To prove this result, we use Lemma 10.

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Lemma 9. Consider the solution \((V_\varepsilon, W_\varepsilon)\) to ODE (28) with the initial condition (35) defined over some interval \([a_0, a]\). We have \(0 < V'_\varepsilon(a) \leq U'_1(ra)\) and \(0 < W'_\varepsilon(a)\). Moreover, if \(V_\varepsilon(a) \geq \tilde{V}(a), W'_\varepsilon(a) < U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra)\).

Proof. Since \(\hat{c}_\varepsilon(a) > ra\) and \(V'_\varepsilon(a) = U'_1(\hat{c}_\varepsilon(a))\), we have \(0 < V'_\varepsilon(a) \leq U'_1(ra)\) due to the concavity of \(U_1\). If \(r \geq \rho\), from Lemma 10, \(W_\varepsilon(a) < \tilde{W}(a)\). Therefore,

\[
W'_\varepsilon(a) = \frac{U_0(\hat{c}_\varepsilon(a)) - \rho W_\varepsilon(a)}{\hat{c}_\varepsilon(a) - ra} > \frac{U_0(ra) - \rho \tilde{W}(a)}{\hat{c}_\varepsilon(a) - ra} = 0.
\]

If \(r < \rho\), Lemma 10 immediately implies that \(W'_\varepsilon(a) > 0\).

To show the upper bound on \(W'_\varepsilon(a)\) when \(V_\varepsilon(a) \geq \tilde{V}(a)\), we use the facts that

\[
(\rho + \lambda) V_\varepsilon(a) - \lambda W_\varepsilon(a) = U_1(\hat{c}_\varepsilon(a)) + V'_\varepsilon(a)(ra - \hat{c}_\varepsilon(a))
\]

and

\[
(\rho + \lambda) \tilde{V}(a) - \lambda \tilde{W}(a) = U_1(ra).
\]

By subtracting the two equalities side by side and rearranging,

\[
\lambda (\tilde{W}(a) - W_\varepsilon(a)) = - (\rho + \lambda) (V_\varepsilon(a) - \tilde{V}(a))
\]

\[
+ U_1(\hat{c}_\varepsilon(a)) - U_1(ra) + V'_\varepsilon(a)(ra - \hat{c}_\varepsilon(a))
\]

\[
\leq U_1(\hat{c}_\varepsilon(a)) - U_1(ra) + V'_\varepsilon(a)(ra - \hat{c}_\varepsilon(a)).
\]

where the last inequality comes from \(V_\varepsilon(a) \geq \tilde{V}(a)\). It follows that

\[
W'_\varepsilon(a) = \frac{U_0(\hat{c}_\varepsilon(a)) - \rho W_\varepsilon(a)}{\hat{c}_\varepsilon(a) - ra} = \frac{U_0(\hat{c}_\varepsilon(a)) - U_0(ra) + \rho (\tilde{W}(a) - W_\varepsilon(a))}{\hat{c}_\varepsilon(a) - ra}
\]

\[
\leq \frac{U_0(\hat{c}_\varepsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} (U_1(\hat{c}_\varepsilon(a)) - U_1(ra) + V'_\varepsilon(a)(ra - \hat{c}_\varepsilon(a)))}{\hat{c}_\varepsilon(a) - ra}
\]

\[
= \frac{U_0(\hat{c}_\varepsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} (U_1(\hat{c}_\varepsilon(a)) - U_1(ra))}{\hat{c}_\varepsilon(a) - ra} - \frac{\rho}{\lambda} V'_\varepsilon(a)
\]

\[
< \frac{U_0(\hat{c}_\varepsilon(a)) - U_0(ra) + \frac{\rho}{\lambda} (U_1(\hat{c}_\varepsilon(a)) - U_1(ra))}{\hat{c}_\varepsilon(a) - ra} + \frac{\rho}{\lambda} \frac{U_1(\hat{c}_\varepsilon(a)) - U_1(ra)}{\hat{c}_\varepsilon(a) - ra} < U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra),
\]

where the last inequality comes from the concavity of \(U_1\) and \(U_0\) and \(\hat{c}_\varepsilon(a) > ra\). \(\square\)

Lemma 10. Consider the solution \((V_\varepsilon, W_\varepsilon)\) to ODE (28) with the initial condition (35) defined over some interval \([a_0, a]\). We have

1) If \(r \geq \rho\), \(W_\varepsilon(a) < \tilde{W}(a)\) \(\forall a > a_0\).
2) If \( r < \rho, W_\epsilon(a) < U_0(\hat{\epsilon}_e(a)) \) \( \forall a > a_0 \).

**Proof.** 1) \( r \geq \rho \): We use Lemma 6 to show property 1). We just need to verify conditions 1) and 2) in Lemma 6. First by definition, \( W_\epsilon(a_0) < \tilde{W}(a_0) \), so condition 1) Lemma 6 is satisfied. For condition 2) in Lemma 6, we show that if \( W_\epsilon(a) = \tilde{W}(a) \), for some \( \forall a > a_0 \) then \( W_\epsilon'(a) < \tilde{W}'(a) \). Indeed,

\[
W_\epsilon'(a) = \frac{U_0(\hat{\epsilon}_e(a)) - \rho W_\epsilon(a)}{\hat{\epsilon}_e(a) - ra} = \frac{U_0(\hat{\epsilon}_e(a)) - \rho \tilde{W}(a)}{\hat{\epsilon}_e(a) - ra} < U_0'(ra).
\]

The last inequality comes from the fact that \( U_0 \) is strictly concave and \( \hat{\epsilon}_e(a) > ra \). On the other hand, we also have

\[
\tilde{W}'(a) = \frac{U_0'(ra)}{U_0'(ra)} \geq U_0'(ra),
\]

because \( r \geq \rho \). Therefore, \( \tilde{W}'(a) > W_\epsilon'(a) \).

2) \( r < \rho \): We also use Lemma 6 to show property 2). By the definition of \( V_\epsilon, W_\epsilon \):

\[
W_\epsilon(a_0) = U_0(ra_0) - \epsilon < U_0(ra_0) < U_0(\hat{\epsilon}_e(a_0)).
\]

So condition 1) in Lemma 6 is satisfied. Now we show that condition 2) in Lemma 6 is also satisfied, i.e. if at some \( a > a_0, W_\epsilon(a) = U_0(\hat{\epsilon}_e(a)) \), we show that \( W_\epsilon'(a) < \frac{d}{da} (U_0'(\hat{\epsilon}_e(a))) \).

Indeed,

\[
W_\epsilon'(a) = \frac{U_0(\hat{\epsilon}_e(a)) - \rho W_\epsilon(a)}{\hat{\epsilon}_e(a) - ra} = 0.
\]

Moreover,

\[
\frac{d}{da} (U_0(\hat{\epsilon}_e(a))) = U_0'(\hat{\epsilon}_e(a)) \hat{\epsilon}_e'(a).
\]

By (33),

\[
\hat{\epsilon}_e'(a) = \frac{1}{U_1''(\hat{\epsilon}_e(a))} \frac{(\rho + \lambda - r) V_\epsilon''(a) - \lambda W_\epsilon'(a)}{ra - \hat{\epsilon}_e(a)}
= \frac{1}{-U_1''(\hat{\epsilon}_e(a))} \frac{(\rho + \lambda - r) U_1'(\hat{\epsilon}_e(a))}{\hat{\epsilon}_e(a) - ra} > 0.
\]

Therefore \( W_\epsilon'(a) = 0 < \frac{d}{da} (U_0'(\hat{\epsilon}_e(a))) \).
D Proof of Theorem 1

Assume $\sigma \neq 1$; the case with $\sigma = 1$ is similar. To proceed, we guess

$$V(a) = \bar{v} \frac{a^{1-\sigma}}{1-\sigma}, \quad W(a) = \bar{w} \frac{a^{1-\sigma}}{1-\sigma},$$

and find $\bar{v}, \bar{w}$ to verify that $V, W$ satisfy (8).\footnote{If $\sigma = 1$}

Given the conjectured functional form, the first-order condition (8a) implies

$$\hat{c}(a) = \psi a,$$

where $\psi = \bar{v}^{\frac{1}{\sigma}}$. Plugging this back into (8a) gives

$$\left(\rho + \lambda\right) \frac{a^{1-\sigma}}{1-\sigma} = \frac{1}{1-\sigma} \left(\left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}}\right)^{1-\sigma} + \left(\bar{v}a^{-\sigma}\right) \left(ra - (\bar{v}a^{-\sigma})^{-\frac{1}{\sigma}}\right) + \lambda \bar{w} \frac{a^{1-\sigma}}{1-\sigma}$$

$$= \frac{\sigma}{1-\sigma} \left(\bar{v}a^{-\sigma}\right)^{-\frac{1}{\sigma}} + \left(\bar{v}a^{-\sigma}\right) ra + \lambda \bar{w} \frac{a^{1-\sigma}}{1-\sigma}.$$
Canceling the $\frac{\sigma^{1-\sigma}}{1-\sigma}$ terms gives

$$\bar{\omega} = \frac{\bar{\beta} \bar{\sigma}^{1-\frac{1}{\sigma}}}{\Delta + (1-\sigma) \bar{\sigma}^{-\frac{1}{\sigma}}}.$$ (44)

Combining equations (42) and (44), we obtain

$$\lambda + \Delta = \sigma \bar{\sigma}^{-\frac{1}{\sigma}} + \lambda \bar{\beta} \bar{\sigma}^{-\frac{1}{\sigma}} \frac{\beta \bar{\sigma}^{-\frac{1}{\sigma}}}{\Delta + (1-\sigma) \bar{\sigma}^{-\frac{1}{\sigma}}},$$

a single equation in $\bar{\sigma}$. Define $\psi \equiv \bar{\sigma}^{-\frac{1}{\sigma}}$. We then have a quadratic equation in $\psi$:

$$P(\psi) \equiv Q_2 \psi^2 + Q_1 \psi + Q_0 = 0,$$ (45)

with

$$Q_2 \equiv (1-\sigma) \sigma$$
$$Q_1 \equiv (\sigma + \bar{\beta} - 1) \lambda + \Delta (2\sigma - 1)$$
$$Q_0 \equiv - (\lambda + \Delta) \Delta.$$

If $\sigma < 1$ then $Q_2 > 0$ and $Q_0 < 0$. This implies that there exists a unique strictly positive $\psi$ that is the solution to (45). The implied consumption rule yields finite utility.

If $\sigma > 1$ we have that $Q_2 < 0$. This implies that

$$P\left(\frac{\Delta}{\sigma - 1}\right) = -\frac{\sigma}{\sigma - 1} \Delta^2 + \left((\sigma + \bar{\beta} - 1) \lambda + \Delta (2\sigma - 1)\right) \frac{\Delta}{\sigma - 1} - (\lambda + \Delta) \Delta$$
$$= -\bar{\beta} \frac{\Delta}{\sigma - 1} \lambda \frac{\Delta}{\sigma - 1} > 0.$$

Therefore, there exists two solutions $0 < \psi_1 < \frac{\Delta}{\sigma - 1} < \psi_2$ such that $P(\psi) = 0$. To know which root corresponds to a solution to (8), we observe that $c_t = \psi a_t$ so $\dot{a}_t = (r - \psi) a_t$ or $a_t = e^{(r-\psi)t} a_0$. Thus $V \propto \int e^{-\rho t} e^{(1-\sigma)(r-\psi)t} dt = \int_0^\infty e^{(-\Delta + \psi(\sigma-1)) t} dt$. For $V$ to finite, we require $\psi < \frac{\Delta}{\sigma - 1}$. So only the smaller root to (45), $\psi_1$, yields finite value functions, and corresponds to a solution to (8).

Now, we turn to the second part of the theorem. Given that $\dot{c}(a) = \psi a$, $\dot{c}(a) < ra$ if and only if $\psi < r$. For $\sigma < 1$ we have $r > 0$ so that $0 < \psi < r$ if and only if $P(r) > P(\psi) = 0$. For $\sigma > 1$, because $\rho > 0$, $r < \frac{\rho - r(1-\sigma)}{\sigma - 1} = \frac{\Delta}{\sigma - 1}$. Given that $P\left(\frac{\Delta}{\sigma - 1}\right) > 0$ and $P(\psi_1) = 0$, $r > \psi$ if and only if $P(r) > P(\psi) = 0$. Thus, we need to establish that $P(r) > 0$. This is
equivalent to
\[ \beta > \frac{\rho}{r} \left( 1 - \frac{r - \rho}{\lambda} \right) = \hat{\beta}. \]
Similarly, \( \hat{c}(a) > ra \), i.e. \( \psi > r \) if and only if \( \bar{\beta} < \hat{\beta} \).

E  Proofs for Dissaving Equilibria

E.1  Proof of Theorem 2

Once we verify the the HJB system (8) all the equilibrium conditions in Subsection 3.1 are met. By Lemma 5, we have 
\[ V'(a) = U'_1(ra), \]
therefore \( \hat{c}(a) = ra \) and equations (8) are satisfied by the definitions of \( V \) and \( W \).

E.2  Proof of Theorem 3

We prove this theorem by construction. Lemma 1 shows that starting from \( a_0 = \bar{a} \), ODE (28) with the boundary condition
\[ (V(a_0), W(a_0)) = (\bar{V}(a_0), \bar{W}(a_0)) \]
admits a solution defined over \([\bar{a}, \bar{a} + \omega)\) for some \( \omega > 0 \). Let \( (V_0, W_0) \) denote this solution. Let \([a, a^*)\) be the right maximal interval of existence for this solution. It is immediate that \( a^* \geq \bar{a} + \omega \). If \( a^* = \infty \), we have found a (continuous) Markov equilibrium, with \( (V, W) = (V_0, W_0) \).

If \( a^* < \infty \), following the steps in the proof of Lemma 8, we can show that
\[ \lim_{a \uparrow a^*} V_0(a) \leq V(a^*). \]
Moreover, as shown in Lemma 1 \( V_0(a) > \bar{V}(a) \) in a neighborhood to the right of \( \bar{a} \). Thus, by the Intermediate Value Theorem, there exists \( a_1 \in (a, a^*) \) such that \( V_0(a_1) = \bar{V}(a_1) \).

Starting from \( a_1 \), we apply Lemma 1 again with \( a_1 \) standing for \( a_0 \) and construct the a solution \( (V_1, W_1) \) to ODE (28) with the boundary condition
\[ (V_1(a_1), W_1(a_1)) = (\bar{V}(a_1), \bar{W}(a_1)). \]
Following this procedure, we obtain a sequence \( a_0 = \bar{a} < a_1 < ... \) with \( \lim_{n \to \infty} a_n = +\infty \) and a sequence of value functions \( (V_n, W_n) \) defined over \([a_n, a_{n+1}]\) with the boundary
condition
\[(V_n(a_n), W_n(a_n)) = (\overline{V}(a_n), \overline{W}(a_n)).\]

The divergence of \{a_n\} is shown in Lemma 11 below.

We define the value and consumption functions \((V, W, \hat{c})\) over the whole interval \([a, \infty)\) as
\[(V(a), W(a), \hat{c}) = (V_n(a), W_n(a), \hat{c}_n(a)) \text{ for } a \in [a_n, a_{n+1}).\]

We verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. Conditions (a)-(e) are satisfied by the construction of \((V, W)\). Condition (f) on the existence of \(a_t\) is satisfied because by construction \(\hat{c}(a)\) is differentiable and \(\hat{c}(a) > ra\) outside steady-states \{\(a_n\)\}. Indeed, if \(a(0) = a_n\) then \(a(t) \equiv a_n\) for all \(t \geq 0\) satisfies ODE (9) for all \(t \geq 0\). If \(a(0) \in (a_n, a_{n+1})\), the solution \(a(t)\) to ODE (9) with the initial condition \(a(t) = a(0)\) exists and is unique over a small interval \([0, c]\) because \(\hat{c}(a)\) is continuously differentiable over \((a_n, a_{n+1})\). In addition, because \(\hat{c}(a) > ra\), \(a(t)\) is strictly decreasing in \(t\). Let \([0, T]\) denote the right maximal interval of existence for \(a(t)\) to ODE (9). If \(T = \infty\), we obtain the existence of \(a(t)\) to ODE (9) over the whole time interval \([0, \infty)\). If \(T < \infty\) (in the Online Appendix we show that this is always the case), by Hartman (2002, Theorem 3.1), \(a(T) = a_n\). Defining \(a(t) = a_n\) for all \(t \geq T\), we also obtain the existence of \(a(t)\) to ODE (9) over the whole time interval \([0, \infty)\). Finally, the limits \(\lim_{t \to \infty} e^{-\rho t} V(a_t)\) and \(\lim_{t \to \infty} e^{-\rho t} W(a_t)\) are both equal to 0 because \(\dot{a}_t \leq 0\), \(V\) and \(W\) are increasing over \([a_n, a_{n+1})\) and \(a_t \geq a_n\).

**Lemma 11.** If the sequence \{\(a_n\)\} constructed in Theorem 3 has an infinite number of elements then

\[
\lim_{n \to \infty} a_n = +\infty.
\]

**Proof.** The result is shown by contradiction. Assume that the sequence is infinite and is bounded above. By construction \(\{a_n\}_{n=0}^\infty\) is strictly increasing, thus the sequence converges to some \(a^\infty\). We assume by contradiction that \(a^\infty\) is finite. By construction, \(V_n(a_n) = \overline{V}(a_n), W(a_n) = \overline{W}(a_n)\) and \(V_n(a_{n+1}) = \overline{V}(a_{n+1})\) and \(V_n(a) > \overline{V}(a)\) for \(a \in (a_n, a_{n+1})\). We can then apply Lemma 9 to show that \(0 \leq V'_n(a) \leq U'_1(ra)\) and \(0 \leq W'_n(a) \leq 36\)

\[
\lim_{t \to \infty} e^{-\rho t} U_1(e^{-\rho t}) \leq 0.
\]

When \(U_1\) is a CRRA function with CRRA coefficient \(\sigma\). This condition is equivalent to \(\rho - (1 - \sigma) r \geq 0\), which is immediately satisfied if \(\sigma \geq 1\) and puts an upper bound on \(r\) if \(\sigma < 1\). This is also a condition which guarantees that the full commitment solution yields finite value.

\[\text{57}\]
\( U'_0(ra) + \frac{\rho}{\lambda} U'_1(ra) \). By the Mean Value Theorem, there exists \( a^*_n \in [a_n, a_{n+1}] \) such that

\[
V'_n(a^*_n) = \frac{V_n(a_{n+1}) - V_n(a_n)}{a_{n+1} - a_n} = \frac{\bar{V}(a_{n+1}) - \bar{V}(a_n)}{a_{n+1} - a_n}.
\]

Since \( \{a_n\} \) converges to \( a^\infty \),

\[
\lim_{n \to \infty} V'_n(a^*_n) = \bar{V}'(a^\infty).
\] (46)

On the other hand \( V'_n(a^*_n) = R_l(a^*_n, V_n(a^*_n), W_n(a^*_n)) \). Since \( \bar{V}(a_n) \leq V_n(a^*_n) \leq \bar{V}(a_{n+1}) \) and \( \bar{W}(a_n) \leq W_n(a^*_n) \leq \bar{W}(a^*_n) \)

\[
= \lim_{n \to \infty} R_l(a^\infty, \bar{V}(a^\infty), \bar{W}(a^\infty)) = U'_1(ra^\infty).
\] (47)

The desired contradiction follows from (46) and (47) which cannot happen at the same time given that \( V'(a^\infty) < U'_1(ra^\infty) \) by condition (34) for \( a = a^\infty \).

\[\square\]

## F Proofs for Saving Equilibria

### F.1 Proof of Theorem 4

We prove this theorem by construction.

Depending on condition (a) or (b) in Assumption 2, we define an asset level \( a_u \), and the value functions \( (V_u, W_u) \) over \([a_u, \infty)\) satisfying the HJB equations (8) as following.

Case 1: Assumption 2a holds.

Without loss of generality we assume that \( \bar{c} \) is the minimum consumption level such that Assumption 2a is satisfied, i.e. \( \beta(c) = \frac{U'_0(c)}{U_0(c)} < 1 \) for all \( c < \bar{c} \) and \( \beta(c) = 1 \) for all \( c \geq \bar{c} \). Therefore there exists \( \bar{u} \) such that \( U_1(c) = U_0(c) + \bar{u} \) for all \( c \geq \bar{c} \). In Subsection A.3.2, we show that there exists \( a_u \) such that \( \hat{c}_0(a) > \bar{c} \) for \( a > a_u \) and \( \hat{c}_0(a_u) = \bar{c} \). For \( a \geq a_u \), let

\[
(V_u(a), W_u(a)) = \left( V_{sp}(a), V_{sp}(a) - \frac{\bar{u}}{\rho + \lambda} \right).
\]

As shown in Subsection A.3.2, \( (V_u, W_u) \) satisfies the HJB equation (8), with \( \hat{c}_u = \hat{c}_0 \). Moreover, \( \hat{c}'_u(a) > 0 \) and, because \( r > \rho \), \( \hat{c}_u(a) < ra \).

Case 2: Assumption 2b holds.
In Theorem 1, we show that for \( a \geq a_u \), \((V_u(a), W_u(a)) = (\phi a^{1-\sigma}, \phi \frac{a^{1-\sigma}}{1-\sigma})\) and \( \hat{c}_u(a) = \phi a \) satisfy the HJB equations (8) over \( [a_u, \infty) \) where and \( a_u = \frac{\xi}{\phi} \). It is immediate that \( \hat{c}_u(a) = \phi > 0 \). Moreover, because \( \tilde{\beta} > \hat{\beta}, \phi < r \), so \( \hat{c}_u(a) < r a \).

Having determined the value and policy functions at and beyond \( a_u \), we construct the value and policy functions below \( a_u \). Noticing that the initial values \((a_u, V_u(a_u), W_u(a_u)) \in E\), a solution \((V_d, W_d)\) to the ODE (29) with the initial condition

\[
(V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))
\]

exists and is unique locally over an interval \((a_u - \epsilon, a_u + \epsilon)\).\(^{37}\) Let \((\hat{a}, a_u + \epsilon)\) denote the (left) maximal interval of existence for this solution. We will show that \( \hat{a} < \frac{\xi}{r} \).

If \( \hat{a} = 0 \), this is immediate. If \( \hat{a} > 0 \), by Lemma 12, \( V_d'(a), W_d'(a) > 0 \) for all \( a > \hat{a} \).

So the limits \( \lim_{a \downarrow \hat{a}} V_d(a) = V_d(\hat{a}) \) and \( \lim_{a \downarrow \hat{a}} W_d(a) = W_d(\hat{a}) \) exist.\(^{38}\) By Hartman (2002, Theorem 3.1), \((\hat{a}, V_d(\hat{a}), W_d(\hat{a})) \in E_\sigma\). Therefore \( \hat{c}_d(\hat{a}) = r \hat{a} \). As shown in Lemma 12, \( V''_d(a) < 0 \). This implies \( \hat{c}_d'(a) = \frac{V''_d(a)}{U_1'(\hat{c}_d(a))} > 0 \). Thus,

\[
r \hat{a} = \hat{c}_d(\hat{a}) < \hat{c}_d(a_u) = \bar{c}.
\]

So \( \hat{a} < \frac{\xi}{r} \).

Given the value and policy functions \((V_u, W_u, \hat{c}_u)\) and \((V_d, W_d, \hat{c}_d)\), for \( a > \hat{a} \), we define the value and policy functions \((V, W, \hat{c})\) over \([\hat{a}, \infty)\) as follows

\[
(V, W, \hat{c}) = \begin{cases} 
(V_d, W_d, \hat{c}_d) & \text{if } a \leq a_u \\
(V_d, W_d, \hat{c}_u) & \text{if } a \geq a_u.
\end{cases}
\]

\(^{37}\)Because of the uniqueness of the solution, \((V_d(a), W_d(a)) = (V_u(a), W_u(a))\) for all \( a \in [a_u, a_u + \epsilon) \).

\(^{38}\)We also show that \( \lim_{a \uparrow \hat{a}} W_d(a) > -\infty \) and \( \lim_{a \downarrow \hat{a}} V_d(a) > -\infty \). This is immediate if \( U_1 \) is bounded from below, and consequently \( U_0 \) is bounded from below by some \( \underline{u} \), because \( \beta(c) \leq 1 \). Because \( W_d'(a) > 0 \) and \( \hat{c}_d(a) < ra \) for \( a > \hat{a} \), (8b) implies that \( W_d(a) > \frac{1}{2} \underline{u} \) for \( a > \hat{a} \). If \( U_1 \) is unbounded from below, we make the additional technical assumption that \( \sigma = \inf_c \sigma(U_1, c) > 1 - \frac{\lambda}{\rho} \). We have

\[
\hat{c}_d'(a) = \frac{(\rho + \lambda - r)V_d'(a) - AW_d'(a)}{U_1'c_d(a)(ra - \hat{c}_d(a))} < \frac{(\rho - r)V_d'(a)}{U_1'(\hat{c}_d(a))(ra - \hat{c}_d(a))}
\]

\[
= \frac{(\rho - r)U_1'(\hat{c}_d(a))(ra - \hat{c}_d(a))}{U_1'(\hat{c}_d(a))(ra - \hat{c}_d(a))} \frac{\hat{c}_d(a)}{\sigma} < \frac{r - \rho}{\sigma} \frac{\hat{c}_d(a)}{ra - \hat{c}_d(a)}
\]

because \( W'(a) < V'(a) \). Therefore \( \hat{c}_d(a) > c^*(a) > 0 \) where \( c^*(a) \) is the solution to the ODE \( c''(a) = \frac{r - \rho}{\sigma} \frac{c(a)}{ra - c(a)} \), and \( c^*(a_u) = \hat{c}_d(a_u) < ra_u \) (the closed form solution for \( c^*(a) \) is a special case of the solution provided in Proposition 6). Therefore, \( \lim_{a \uparrow \hat{a}} W_d(a) > \frac{1}{\rho} U_0(c^*(\hat{a})) > -\infty \) for all \( a \geq \hat{a} \). Finally, \( \lim_{a \downarrow \hat{a}} V_d(a) > \frac{\hat{c}_d(a)U_0(c^*(\hat{a})) + U_1(\hat{a})}{\rho + \lambda} > -\infty \).
As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. In addition, \( \hat{c}(a) < ra \) and \( \hat{c}'(a) > 0 \).

F.2 Supporting Results for Proof of Theorem 4

Lemma 12. Assume \( \rho < r \). Consider a solution \((V, W)\) to ODE (29) defined over \((\hat{a}, a_u]\) with the initial condition \((V(a_u), W(a_u) = (V_u(a_u), W_u(a_u))\) with \(a_u, V_u, W_u\) defined in Subsection F.1 (depending on Assumption 2a or 2b). Then for all \( a < a_u \)

1) \( (\rho + \lambda - r) V'(a) - \lambda W'(a) < 0 \) and \( W'(a) > 0 \)
2) \( V''(a) < 0 \)
3) \( V'(a) > W'(a) \).

Proof. We prove this lemma in two steps. Step 1: If properties 1),2) and 3) hold in a neighborhood to the left of \( a_u \) then they hold for all \( a < \hat{a} \). Step 2: Verify that properties 1),2) and 3) hold in a neighborhood to the left of \( a_u \) separately under Assumption 2a or 2b.

Step 1: Assume that properties 1),2) and 3) hold in a neighborhood to the left of \( a_u \). We show that 1),2), and 3) hold for all \( a < a_u \).

We prove 1) separately for two cases: Case 1: \( \rho + \lambda - r > 0 \) and Case 2: \( \rho + \lambda - r \leq 0 \).

Case 1: By construction, \( V'(a) = U'_1(\hat{c}(a)) > 0 \). Therefore if \( (\rho + \lambda - r) V'(a) - \lambda W'(a) < 0 \), \( W'(a) > 0 \). We just need to show the first inequality.

We prove this inequality using Lemma 6 (Variation 1). Condition 1) of Lemma 6 (at \( a_u \)) is satisfied by assumption. We just need to verify Condition 2) of Lemma 6, i.e. if there exists \( \bar{a} < a_u \) such that

\[
(\rho + \lambda - r) V'(\bar{a}) - \lambda W'(\bar{a}) = 0. \tag{48}
\]

then

\[
(\rho + \lambda - r) V''(\bar{a}) > \lambda W''(\bar{a}).
\]

By Lemma 4,

\[
(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a) (ra - \hat{c}(a)). \tag{49}
\]

At \( a = \bar{a} \), because of (48), and \( r\bar{a} > \hat{c}(\bar{a}) \), \( V''(\bar{a}) = 0 \).

Differentiating the second equation, (8b) in system (8), and using (33), we have

\[
pW'(a) = \frac{U'_0(\hat{c}(a))}{U''_1(\hat{c}(a))} V''(a)
+ W''(a) (ra - \hat{c}(a)) + W'(a) \left( r - \frac{1}{U''_1(\hat{c}(a))} V''(a) \right). \tag{50}
\]

60
At $a = \bar{a}$, using the previous result that $V''(\bar{a}) = 0$, and rearranging, we arrive at

$$W''(\bar{a}) (r\bar{a} - \hat{c}(\bar{a})) = (\rho - r) W'(\bar{a}).$$

Because, $W(\bar{a}) = \frac{(\rho + \lambda - r)V'(\bar{a})}{\lambda} > 0$ and $\rho - r < 0$, the right hand side is strictly negative. Moreover $ra - \hat{c}(a) > 0$, therefore $W''(\bar{a}) < 0$. Thus,

$$W''(\bar{a}) < 0 = V''(\bar{a}),$$

i.e. we have verified the second condition in Lemma 6. Given that the two conditions of Lemma 6 are satisfied, this lemma implies the first property.

**Case 2:** Because $\rho + \lambda \leq r$ and $V'(a) > 0$, $(\rho + \lambda - r) V'(a) - \lambda W'(a) < 0$ if $W'(a) > 0$. Therefore we just need to show the last inequality. Again we prove this inequality using Lemma 6 (Variation 1). Condition 1) of Lemma 6 at $a_u$ is shown in the proof of Theorem 4. We now verify Condition 2). If there exists $\bar{a} < a_u$ such that $W'(\bar{a}) = 0$ we show that $W''(\bar{a}) < 0$. From equation (49) at $\bar{a}$, $(\rho + \lambda - r) V'(\bar{a}) = V''(\bar{a}) (r\bar{a} - \hat{c}(\bar{a})).$ This implies $V''(\bar{a}) < 0.$ From (50), since $W'(\bar{a}) = 0$,

$$0 = \frac{U_0(\hat{c}(\bar{a}))}{U_1(\hat{c}(\bar{a}))} V''(\bar{a}) + W''(\bar{a}) (r\bar{a} - \hat{c}(\bar{a})).$$

Therefore $W''(\bar{a}) < 0.$ Given that the two conditions of Lemma 6 are satisfied, this lemma implies $W'(a) > 0$ for all $a$.

The second property immediately follows using (49) and $ra - \hat{c}(a) > 0$.

We also prove the third property similarly by using Lemma 6. Condition 1) in Lemma 6 is satisfied. We now verify that condition 2) is also satisfied. Indeed, if there exists $\bar{a} < a_u$ such that $W'(\bar{a}) = V'(\bar{a}).$ By (49), at $a = \bar{a}$,

$$V''(\bar{a}) = \frac{(\rho - r)V'(\bar{a})}{r\bar{a} - \hat{c}(\bar{a})} < 0.$$

Again by equation (50),

$$W''(\bar{a}) = (\rho - r) W'(\bar{a}) - \left(\frac{U_0(\hat{c}(\bar{a}))}{U_1(\hat{c}(\bar{a}))} - W'(\bar{a})\right) \frac{1}{U_1''(\hat{c}(\bar{a}))} V''(\bar{a}) < 0.$$

The second line comes from the assumption that $W'(\bar{a}) = V'(\bar{a}) = U_1'(\hat{c}(\bar{a})) > U_0'(\hat{c}(\bar{a}))$.
by contradiction that (such that Because So by Lemma 6, we obtain the third property.

Step 2: We show that properties 1),2) and 3) hold in a neighborhood to the left of \( a_u \). We treat the two cases associated with Assumption 2a or 2b separately.

Under Assumption 2b, given the closed form solution given in Appendix D, we show that properties 1), 2), 3) are satisfied at \( a_u \) in Lemma 13 below. By continuity, properties 1), 2), 3) hold in a neighborhood to the left of \( a_u \).

Under Assumption 2a. It is easy to verify that properties 1) and 2) are satisfied because \( V_{sp}(a) \) is strictly concave and \( W_{sp}'(a) = V'_u(a) = V_{sp}'(a) \) for \( a \geq a_u \), and thus \( V'(a_u) = W'(a_u) = V_{sp}'(a_u) > 0 \) and \( V''(a_u) = V_{sp}''(a_u) < 0 \). By continuity properties 1) and 2) hold in a neighborhood to the left of \( a_u \). Properties 3) is not satisfied at \( a_u \) because \( V''(a_u) = W''(a_u) \) so \( V'(a) < W'(a) \) in some neighborhood to the left of \( a_u \).

Indeed, consider the solution \((V_e, W_e)\) to ODE (29) with the initial condition

\[
(V_e(a_u), W_e(a_u)) = \left( V_u(a_u) - \frac{\epsilon}{\rho + \lambda}, W_u(a_u) - \frac{\epsilon}{\lambda} \right).
\]

Because \((a_u, V_u(a_u), W_u(a_u)) \in E\), by Hartman (2002, Theorem 3.2), there exists \( \omega > 0 \) such that \((V_e, W_e)\) are defined over, \([a_u - \omega, a_u]\) and \((V_e, W_e) \rightarrow (V, W)\) uniformly over \([a_u - \omega, a_u]\) as \( \epsilon \rightarrow 0 \).

It easy to verify that \( V'_u(a_u) > W'_e(a_u) \). Because, from the initial conditions, we have \( \hat{c}_e(a_u) = \hat{c}_u(a_u) \), and \( V'_u(a_u) = U'_1(\hat{c}_u(a_u)) = U'_1(\hat{c}_e(a_u)) = V'_e(a_u) \),

\[
W'_e(a_u) = \frac{\rho \left( W_u(a_u) - \frac{\epsilon}{\lambda} \right) - U_0(\hat{c}_u(a_u))}{ra_u - \hat{c}_u(a_u)} < \frac{\rho W_u(a_u) - U_0(\hat{c}_u(a_u))}{ra_u - \hat{c}_u(a_u)} = W'_u(a_u) = V'_u(a_u) = V'_e(a_u).
\]

In addition, when \( \epsilon \) sufficiently small, we also have properties 1) and 2) holds for \( V_e, W_e \) at \( a_u \). Therefore, following the proofs in Step 1 above, we can show that properties 1),2),3) hold for all \( a \in [a_u - \omega, a_u] \) for \((V_e, W_e)\). In particular, \( V'_e(a) > W'_e(a) \) for all \( a \in [a_u - \omega, a_u] \).

Now as \( \epsilon \rightarrow 0 \), \((V_e, W_e) \rightarrow (V, W)\). So \( V'(a) \geq W'(a) \) for all \( a \in [a_u - \omega, a_u] \). We show by contradiction that \( V'(a) > W'(a) \) for all \( a \in (a_u - \omega, a_u) \). Assume to the contrary that \( V'(\tilde{a}) = W'(\tilde{a}) \) for some \( \tilde{a} < a_u \). As in Step 1, this implies that \( V''(\tilde{a}) < W''(\tilde{a}) \) strictly,
because \( \beta(\check{c}(a)) < 1 \) given that \( \check{c}(a) < \check{c}(a_u) = \check{c} \). Therefore in the right neighborhood of \( \check{a} \), \( V'(a) < W'(a) \), which contradicts the earlier result that \( V'(a) \geq W'(a) \). Thus, \( V'(a) > W'(a) \) for all \( a \in (a_u - \omega, a_u) \).

\[ \square \]

**Lemma 13.** The linear equilibria in Theorem 1 with \( \hat{\beta} < \bar{\beta} < 1 \) satisfies, for all \( a > 0 \)

1) \((\lambda + \rho - r) V'(a) < \lambda W'(a) \)
2) \(V''(a) > 0 \)
3) \(W'(a) < V'(a) \)

**Proof.** As shown in Theorem 1, because \( \bar{\beta} > \hat{\beta} \), \( \check{c}(a) < ra \). By Lemma 4,

\[
(\rho + \lambda - r) V'(a) - \lambda W'(a) = V''(a) (ra - \check{c}(a)).
\]

Because \( V''(a) = -\bar{\sigma} a^{-\sigma - 1} < 0 \), and \( \check{c}(a) < ra \),

\[
(\lambda + \rho - r)V'(a) - \lambda W'(a) < 0.
\]

The second inequality 2) is immediate because \( V''(a) = -\bar{\sigma} a^{-\sigma - 1} < 0 \).

The last inequality 3) is equivalent to \( \frac{\hat{\beta} \psi}{\Delta + (1-\sigma) \psi} < 1 \). After algebra manipulation, this inequality holds if and only if \((\sigma + \bar{\beta} - 1) \psi < \Delta \). This obviously holds if \( \sigma + \bar{\beta} \leq 1 \), because \( \Delta, \psi > 0 \). If \( \sigma + \bar{\beta} > 1 \), we show that \( \psi < \frac{\Delta}{\sigma + \bar{\beta} - 1} \). Indeed, \( P\left(\frac{\Delta}{\sigma + \bar{\beta} - 1}\right) = \frac{\hat{\beta}(1-\bar{\beta})}{(\sigma + \bar{\beta} - 1)^2} > 0 \). Therefore, \( \psi < \frac{\Delta}{\sigma + \bar{\beta} - 1} \).

\[ \square \]

**G  Proofs for Poverty Trap Equilibria**

**G.1  Proof of Theorem 5**

We prove this theorem by construction.

As shown in Theorem 4 there exists a unique solution \((V_d, W_d)\) to ODE (29) that satisfies \((V_d(a_u), W_d(a_u)) = (V_u(a_u), W_u(a_u))\) (with \(a_u, V_u, W_u\) defined differently under Assumption 2a or 2b) defined over a maximal interval of existence \((\check{a}, a_u]\), where \( \check{a} < \frac{\xi}{\hat{\rho}} \).

We also show in Theorem 4 that \( V_d, W_d \) is defined and is continuous at \( \check{a} \), i.e. the limits \( \lim_{a \uparrow \check{a}} V_d(a) = V_d(\check{a}) \) and \( \lim_{a \downarrow \check{a}} W_d(a) = W_d(\check{a}) \) exist, and \( \check{c}_d(\check{a}) = r \check{a} \).

If \( \check{a} \leq a \), let

\[
(V, W, \check{c}) = \begin{cases} (V_d, W_d, \check{c}_d) & \text{if } a \leq a < \check{a} \\ (V_u, W_u, \check{c}_u) & \text{if } a \geq \check{a}. \end{cases}
\]
If \( a < \bar{a} \), Lemma 14 below shows that \( V_d(\bar{a}) = \overline{V}(\bar{a}) \). In addition, Lemma 15 shows that \( V_d(a) > \overline{V}(a) \) for all \( a > \bar{a} \). Therefore,

\[
V_d'(\bar{a}) = U_1'(r\bar{a}) \geq \overline{V}'(\bar{a}).
\]

By Lemma 5, \( \beta(r\bar{a}) \leq \hat{\beta} \).

If \( \beta(r\bar{a}) = \hat{\beta} \), let \( \bar{a} = \min \{ a \geq a : \beta(ra) = \hat{\beta} ) \}. Because \( \beta \) is weakly increasing, \( \beta(ra) = \hat{\beta} \) for all \( a \in [\bar{a}, \bar{a}] \). We define \( V_h, W_h \) over \([\bar{a}, \infty)\) such that

\[
(V_h(a), W_h(a), \hat{c}_h) = \begin{cases} 
(V_u(a), W_u(a), \hat{c}_u(a)) & \text{if } a_u \leq a \\
(V_d(a), W_d(a), \hat{c}_d(a)) & \text{if } \bar{a} \leq a < a_u \\
(\overline{V}(a), \overline{W}(a), ra) & \text{if } \bar{a} \leq a < \bar{a}.
\end{cases}
\]

By Theorems 2 and 4, \( V_h, W_h \) satisfy (8) over \([\bar{a}, \infty)\). Replacing \( \hat{a} \) by \( \bar{a} \) if \( \beta(r\hat{a}) = \hat{\beta} \), without loss of generality, we can assume that \( \beta(ra) < \hat{\beta} \) for \( a < \bar{a} \).

Iteratively, we construct a sequence \( \{ a_i \} \) starting with \( a_0 = \bar{a} \), and for each \( i \geq 0 \), \( a_i < \bar{a} \) and \( \beta(ra_i) < \hat{\beta} \) and the value and policy functions \((V_i, W_i, \hat{c}_i)\) are determined as following:

Iteration \( i \): Starting from \( a_i < \bar{a} \), because \( \beta(ra_i) < \hat{\beta} \), using Lemma 1, we show that ODE (28) admits a solution \((V_i, W_i)\), with the initial condition \((V_i(a_i), W_i(a_i)) = (\overline{V}(a_i), \overline{W}(a_i))\), defined over a (right) maximal interval of existence \([a_i, a_i^+]\). Moreover \( V_i(a) > \overline{V}(a) \) in a neighborhood to right of \( a_i \). There are three possibilities:

i-1) \( a_i^+ < \bar{a} \). Then following the steps in Lemma 8, we can shows that \( V_i(a_i^+) \leq \overline{V}(a_i^+) \).

By the Intermediate Value Theorem, there exists \( a_i < a_{i+1} < \bar{a} \), such that \( V_i(a_{i+1}) = \overline{V}(a_{i+1}) \) and \( V_i(a) > \overline{V}(a) \) for \( a \in (a_i, a_{i+1}) \). Because \( a_{i+1} < \bar{a} \), \( \beta(ra_{i+1}) < \hat{\beta} \). Go to iteration \( i+1 \) with \( a_{i+1} \) standing for \( a_i \).

i-2) \( a_i^+ \geq \bar{a} \) and \( V_i(a) \leq \overline{V}(a) \) for some \( a < \bar{a} \). By the Intermediate Value Theorem, there exists \( a_i \leq a_{i+1} < \bar{a} \), such that \( V_i(a_{i+1}) = \overline{V}(a_{i+1}) \) and \( V_i(a) > \overline{V}(a) \) for \( a \in (a_i, a_{i+1}) \). Go to iteration \( i+1 \) with \( a_{i+1} \) standing for \( a_i \).

i-3) \( a_i^+ \geq \bar{a} \) and \( V_i(a) > \overline{V}(a) \) for all \( a < \bar{a} \). We stop the construction.

Following this procedure, we produce a strictly increasing sequence \( \{ a_i \} \) such that for each \( i \geq 0 \), \( a_i < \bar{a} \) and \( \beta(ra_i) < \hat{\beta} \) and the value functions \((V_i, W_i)\) satisfies \((V_i(a_i), W_i(a_i)) = (\overline{V}(a_i), \overline{W}(a_i))\) and \( V_i(a_{i+1}) = \overline{V}(a_{i+1}) \) and \( V_i(a) > \overline{V}(a) \) for all \( a \in (a_i, a_{i+1}) \). Let

\[
(V_i(a), W_i(a), \hat{c}_i) = (V_i(a), W_i(a), \hat{c}_i(a)) \quad \text{for } a \in [a_i, a_{i+1}),
\]

with \( a_{n+1} = a_n^+ \) if possibility n-3) is reached at some iteration \( n \).
There are two possible cases:

Case 1: The sequence \{a_i\} is finite, i.e. possibility n-3) is reached at some iteration \(n\):
We obtain a sequence \(a_0 < a_1 < \ldots < a_n < \hat{a}\).

If \(a_n^* = \infty\), we define the value and consumption functions \((V, W, \hat{c})\) over the whole interval \([\hat{a}, \infty)\) as
\[
(V(a), W(a), \hat{c}(a)) = (V_l(a), W_l(a), \hat{c}_l(a)).
\]

If \(a_n^* < \infty\). Following the steps in Lemma 8, we can show that \(V_n(a_n^*) \leq \overline{V}(a_n^*)\).

Therefore, both \(V_n\) and \(V_h\) are defined over \([\hat{a}, a_n^*]\) and
\[
V_h(\hat{a}) = \overline{V}(\hat{a}) \leq V_n(\hat{a})
\]
\[
V_h(a_n^*) \geq \overline{V}(a_{n+1}^*) \geq V_n(a_{n+1}^*).
\]

By the Intermediate Value Theorem, there exists \(a^* \in [\hat{a}, a_n^*]\) such that
\[
V_n(a^*) = V_h(a^*).
\]

We define \((V, W, \hat{c})\) as
\[
(V, W, \hat{c}) = \begin{cases} 
(V_l, W_l, \hat{c}_l) & \text{if } a < a^* \\
(V_h, W_h, \hat{c}_h) & \text{if } a \geq a^*. 
\end{cases}
\]

Case 2: The sequence \{a_i\} is infinite (possibility i-3 is never reached). Then \(\lim_{i \to \infty} a_i = a_\infty \leq \hat{a}\) and \(\beta(ra_\infty) = \hat{\beta}\). Because \(\beta(ra) < \hat{\beta}\) for \(a < \hat{a}\). We have \(a_\infty = \hat{a}\). In this case
\[
(V, W, \hat{c}) = \begin{cases} 
(V_l, W_l, \hat{c}_l) & \text{if } a < a_\infty = \hat{a} \\
(V_h, W_h, \hat{c}_h) & \text{if } a \geq \hat{a}. 
\end{cases}
\]

In all cases we can construct the value and policy functions \((V, W, \hat{c})\) over \([\hat{a}, \infty)\). As in the proof of Theorem 3, we can verify that this construction satisfies all the conditions in Subsection 3.1 for a Markov equilibrium. In addition, \(\hat{c}(a) < ra\) for all \(a \geq a^*\) and \(\hat{c}(a) \geq ra\) for all \(a < a^*\).

G.2 Supporting Results for Theorem 5

Lemma 14. Given the definition of \(V_d\) and \(\hat{a}\) in the of Theorem 5, if \(\hat{a} > 0\) then \((V_d(\hat{a}), W_d(\hat{a})) = (\overline{V}(\hat{a}), \overline{W}(\hat{a}))\).
Because we first show that into the first equation, we obtain inequality.

**Lemma 15.** Given the definition of $V_{d}^\prime(a)$ as

$$W_{d}^\prime(a) = \frac{\rho W_{d}(a) - U_{0}(\hat{c}_{d}(a))}{ra - \hat{c}_{d}(a)} \quad \rightarrow \quad \frac{\rho W_{d}(\hat{a}) - \rho U_{0}(\hat{r}_d)}{\hat{r}_d - \hat{r}_d} = +\infty$$

as $a$ approaches $\hat{a}$ from the right because $\hat{c}_{d}(a) \rightarrow \hat{r}_d$. Moreover, by the continuity of $R_{l}$, $\lim_{a \rightarrow \hat{a}} V_{d}^\prime(a) = U_{l}^\prime(\hat{r}_d)$. This contradicts the property 3) in Lemma 12 that $W_{d}^\prime(a) < V_{d}^\prime(a)$ for all $a > \hat{a}$. Therefore, $W_{d}(\hat{a}) \leq \overline{W}(\hat{a})$.

We also show that $W_{d}(\hat{a}) \geq \overline{W}(\hat{a})$. Assume by contradiction that, $W_{d}(\hat{a}) < \overline{W}(\hat{a})$, then, similarly to the previous case,

$$W_{d}^\prime(a) = \frac{\rho W_{d}(a) - U_{0}(\hat{c}_{d}(a))}{ra - \hat{c}_{d}(a)} \quad \rightarrow \quad \frac{\rho W_{d}(\hat{a}) - \rho U_{0}(\hat{r}_d)}{\hat{r}_d - \hat{r}_d} = -\infty$$

as $a$ approaches $\hat{a}$ from the right. This contradicts the property 1) in Lemma 12 that $W_{d}^\prime(a) > 0$ for all $a > \hat{a}$. Therefore, $W_{d}(\hat{a}) \geq \overline{W}(\hat{a})$.

The two results imply that $W_{d}(\hat{a}) = \overline{W}(\hat{a})$. Combining this equality with the fact that $(\hat{a}, V_{d}(\hat{a}), W_{d}(\hat{a})) \in E_{s}$ yields $V_{d}(\hat{a}) = \overline{W}(\hat{a})$. 

**Lemma 15.** Given the definition of $V_{d}$ and $\hat{a}$ in the of Theorem 5, $V_{d}(a) > \overline{V}(a)$ for all $a > \hat{a}$.

**Proof.** Let $\tilde{U}(c) \equiv U_{1}(c) + \frac{\lambda}{\rho} U_{0}(c)$. By the concavity of $U_{1}$ and $U_{0}$, $\tilde{U}$ is also strictly concave. We first show that $ra > c_{d}(a) > c^\ast(a)$ where $c^\ast(a)$ is defined by

$$\tilde{U}^\prime(c^\ast(a)) = V^\prime(a) + \frac{\lambda}{\rho} W^\prime(a).$$

Indeed, because $\tilde{U}$ is strictly concave, this is equivalent to $\tilde{U}^\prime(c^\ast(a)) > \tilde{U}^\prime(\hat{c}_{d}(a))$ or

$$V_{d}^\prime(a) + \frac{\lambda}{\rho} W_{d}^\prime(a) > U_{l}^\prime(\hat{c}_{d}(a)) + \frac{\lambda}{\rho} U_{0}^\prime(\hat{c}_{d}(a)).$$

Because $V_{d}^\prime(a) = U_{l}^\prime(\hat{c}_{d}(a))$ and $W_{d}^\prime(a) > U_{0}^\prime(\hat{c}_{d}(a))$ by Lemma 16, we obtain the desired inequality.

Now using system (8), substituting $W_{d}$ by the right hand side of the second equation into the first equation, we obtain

$$(\rho + \lambda)V_{d}(a) = \overline{U}(\hat{c}(a)) + (V_{d}^\prime(a) + \frac{\lambda}{\rho} W_{d}^\prime(a))(ra - \hat{c}(a)).$$
Let
\[ F(a, c) \equiv \tilde{U}(c) + (V_\ell'(a) + \frac{\lambda}{\rho} W_d'(a))(ra - c). \]

Because \( \tilde{U} \) is strictly concave, \( F \) is strictly concave in \( c \). By the definition of \( \tilde{U} \) and \( c^*(a) \),
\[
\frac{\partial F(a,c)}{\partial c} = 0 \quad \text{at} \quad c = c^*(a) \quad \text{and} \quad \frac{\partial F(a,c)}{\partial c} < 0 \quad \text{for} \quad c > c^*(a). \]

Therefore
\[
F(a,c^*(a)) > F(a,\hat{c}(a)) > F(a,ra) = (\rho + \lambda)\tilde{V}(a).
\]

Moreover, \( F(a,\hat{c}(a)) = (\rho + \lambda)V_d(a) \), so \( V_d(a) > \tilde{V}(a) \). \qed

**Lemma 16.** Given the definition of \( W_d \) and \( \hat{a} \) in the of Theorem 5, \( W_d'(a) > U_0'(\hat{c}(a)) \) for all \( a \in (\hat{a},a_u) \).

**Proof.** Assumption 3 is equivalent to

\[
\frac{-U_0''(c)}{U_0'(c)} \leq \frac{-U_0''(c)}{U_1'(c)}
\]

for all \( c \leq \tilde{c} \). We use Lemma 6 to prove this lemma. Indeed, we first show that condition 2) in Lemma 6 is satisfied, i.e. if \( W_d'(a) = U_0'(\hat{c}(a)) \) then

\[
\frac{d}{da}(W_d'(a)) < \frac{d}{da}(U_0'(\hat{c}(a))).
\]

Indeed, differentiating equation (8b) with respect to \( a \) implies

\[
\rho W_d'(a) = U_0'(\hat{c}(a))\hat{c}_d'(a) + W_d''(a)(ra - \hat{c}_d(a)) + W_d'(a)(r - \hat{c}_d'(a)).
\]

Because \( W_d'(a) = U_0'(\hat{c}(a)) \), this equation simplifies to

\[
W_d''(a) = \frac{(r - \hat{c}_d(a))}{ra - \hat{c}_d(a)} W_d'(a).
\]

On the other hand,

\[
\frac{d}{da}(U_0'(\hat{c}_d(a))) = U_0''(\hat{c}_d(a))\hat{c}_d'(a) = U_0''(\hat{c}_d(a))\frac{V_d''(a)}{U_1'(\hat{c}_d(a))} = U_0''(\hat{c}_d(a))\frac{(r - \hat{c}_d(a))}{ra - \hat{c}_d(a)},
\]

which implies

\[
\frac{d}{da}(U_0'(\hat{c}_d(a))) \geq U_0''(\hat{c}_d(a))\frac{(r - \hat{c}_d(a))}{U_1''(\hat{c}_d(a))(ra - \hat{c}_d(a))}. \quad (53)
\]
where the last inequality comes from $U_*^i(\hat{c}_d(a)) \geq U_*^0(\hat{c}_d(a))$. Combining this with $\rho < r$ and condition (51), we have (52), but with weak inequality. Now we show that it holds with strict inequality. For $a < a_u$, because $\hat{c}'_d > 0$, $\hat{c}_d(a) < \hat{c}_d(a_u) = \bar{c}$, therefore $U_*^i(\hat{c}_d(a)) > U_*^0(\hat{c}_d(a))$ (this also holds for $a = a_u$ under Assumption 2b). Thus (53) holds with strict inequality. If $a = a_u$ and under Assumption 2a, $\hat{c}_d(a) = \bar{c}$, (51) holds with strict inequality (we assume that $U_*^i(c) > U_*^0(c)$ for $c < \bar{c}$). Hence, in either case, (52) holds with strict inequality.

Now, we show that condition 1) in Lemma 6 is also satisfied. Under Assumption 2b of Theorem 4 with power utility, it is shown in Lemma 17 that at $a_u$ that $W_*^d(a_u) > U_*^0(\hat{c}_d(a_u))$. Under Assumption 2a of Theorem 4, $U_*^i(c) = U_*^0(c)$ for $c \geq \bar{c}$, given how $a_u$ is defined in Subsection A.3.2, we have $W_*^d(a_u) = V_*^d(a_u) = U_*^i(c) = U_*^0(c)$, so $W_*^d(a) = U_*^0(\hat{c}(a))$ at $a = a_u$. Therefore, by (52), $W_*^d(a) > U_*^0(\hat{c}_d(a))$ in the left neighborhood of $a_u$.$^{39}$

Given that both conditions in Lemma 6 are satisfied, it implies that $W'(a) > U_*^0(\hat{c}(a))$ for all $a \in (\hat{a}, a_u)$.

**Lemma 17.** The linear equilibria described in Theorem 1 with $\hat{\beta} < 1$ satisfies $W'(a) > U_*^0(\hat{c}(a))$ for all $a > 0$.

**Proof.** From Theorem 1, $W'(a) > U_*^0(\hat{c}(a))$ is equivalent to

$$\frac{\bar{\beta} \bar{\sigma}^{1 - \frac{1}{\bar{\sigma}}} - \bar{\rho}}{\Delta + \left(1 - \sigma\right) \bar{\sigma}^{- \frac{1}{\bar{\sigma}}}} > \bar{\beta} \bar{\sigma},$$

or equivalently $\psi > \frac{A}{\bar{c}}$. This inequality holds because $P\left(\frac{A}{\bar{c}}\right) = (\bar{\beta} - 1)\frac{\lambda}{\bar{c}} < 0$.

$^{39}$ Another way to show this is to proceed as in the proof of Theorem 3 by considering the solution $(V_\epsilon, W_\epsilon)$ to the ODE (29) with the initial condition $(V_\epsilon(a_u), W_\epsilon(a_u)) = \left(V_u(a_u) + \frac{\bar{\epsilon}}{\bar{\rho} + \epsilon}, W_u(a_u) + \frac{\bar{\epsilon}}{\bar{\rho}}\right)$. It is easy to verify that $W_*^d(a_u) > U_*^0(\hat{c}_d(a_u))$ because $\hat{c}_d(a_u) = \bar{c}(a_u)$. Therefore by Lemma 16, $V_*^d(a) > W_*^d(a)$ for all $a < a_u$. As $\epsilon \to 0$, $(V_\epsilon, W_\epsilon) \to (V_\delta, W_\delta)$. As a result, $W_*^d(a) > U_*^0(\hat{c}_d(a))$ for all $a < a_u$. We can then apply Lemma 16 to show that $W_*^d(a) > U_*^0(\hat{c}_d(a))$ for all $a < a_u$ because $\beta(c) < 1$ for all $c < \bar{c}$, which implies that (52) holds with strict inequality.

**G.3 Proof of Proposition 3**

Fixing $\hat{a} \in (0, \frac{\bar{\epsilon}}{\bar{\rho}})$. First we show that there exists $r_1$ such that for $r < r_1$ such that

$$V(a_u) < \int_{\hat{a}}^{a_u} U_*^i(ra) \, da + \bar{V}(\hat{a}).$$

(54)
Indeed, \( \int_{\bar{a}}^{a_u} U'_1(ra)da = \frac{1}{r} U_1(ra_u) - \frac{1}{r} U_1(r\bar{a}). \) When \( r \to \rho, V(a_u) \to \frac{1}{\rho + \lambda} \left( U_1(\bar{c}) + \frac{\lambda}{\rho} U_0(\bar{c}) \right) \) and \( \frac{1}{r} U_1(ra_u) \to \frac{1}{\rho} U_1(\bar{c}). \) The right hand side of (54) converges to

\[
\frac{1}{\rho} U_1(\bar{c}) - \frac{1}{\rho} U_1(\rho\bar{a}) + \frac{1}{\rho + \lambda} (U_1(\rho\bar{a}) + \frac{\lambda}{\rho} U_0(\rho\bar{a})),
\]

Therefore, at the limit \( r \to \rho, (54) \) is equivalent to

\[
U_0(\bar{c}) - U_1(\bar{c}) < U_0(\rho\bar{a}) - U_1(\rho\bar{a}) \tag{55}
\]

By Mean Value Theorem, there exists \( \bar{c} \in (\rho\bar{a}, \bar{c}) \) such that

\[
\frac{U_0(\bar{c}) - U_0(\rho\bar{a})}{U_1(\bar{c}) - U_1(\rho\bar{a})} = \frac{U'_0(\bar{c})}{U'_1(\bar{c})} = \beta(\bar{c}) < \beta(\bar{c}) = 1,
\]

which is equivalent to (55). So, by continuity, (54) holds when \( r \) belongs to a neighborhood to the right of \( \rho, [\rho, r_1]. \)

Under (54), we show by contradiction that \( \hat{a} > \bar{a}. \) Assume \( \bar{a} \geq \hat{a}. \) By Lemma 15, \( V(\bar{a}) \geq V(\bar{a}). \) Because \( \hat{c}(a) < ra \) for \( \bar{a} < a < a_u, V'(a) > U'_1(ra). \) So

\[
V(a_u) - V(\bar{a}) > \int_{\bar{a}}^{a_u} U'_1(ra)da,
\]

which contradicts (54). So \( \hat{a} > \bar{a} > 0. \) Now pick any \( \tilde{a} \) such that \( 0 < \tilde{a} < \bar{a}. \) We have \( \hat{a} > \tilde{a}. \)

The construction in Theorem 5 implies that \( a^* > \bar{a}. \) It remains to show that \( a^* < \infty. \) More strongly, we show by contradiction that \( a^* < a_u. \) Assume \( a^* \geq a_u. \) By the definition of \( a_u, V_u(a) = V_{sp}(a) \) for all \( a \geq a_u. \) Therefore,

\[
V_u(a^*) = V_{sp}(a^*) > V_l(a^*),
\]

which contradicts the definition of \( a^* \) that \( V_u(a^*) = V_l(a^*). \)

H Proofs for Convergence Equilibria

Proof of Theorem 6. We can show the existence of Markov equilibrium following the steps of the proofs of Lemma 1 and Proposition 4 below. In particular, we show that ODEs (29) (and (28)) admit a solutions over some neighborhood to the left (and right) of \( \bar{a}, \) with
the initial condition \((V(\bar{a}), W(\bar{a})) = (\bar{V}(\bar{a}), \bar{W}(\bar{a}))\) by taking the limit of a sequence of solutions to ODEs \((29)\) (and \((28)\)) starting at non-singular initial conditions at \(\bar{a}\). However, in this (more informal) proof, we present an intuitive argument.

We look for an equilibrium defined in a local neighborhood of \(\bar{a}\) with \(\dot{c}(a) = r\bar{a} + \Psi(a - \bar{a}) + o(a - \bar{a})\). For stability, we require that \(\Psi > r\). This implies that around \(\bar{a}\), \(\dot{a}_t \approx (r - \Psi)a_t\), therefore, \(a_t \approx (a_0 - \bar{a})e^{(r-\Psi)t} + \bar{a}\). Now

\[
W(a_t) = \int_0^\infty e^{-\rho s}U_0(c_{t+s})ds
\]

\[
\approx \frac{1}{\rho}U_0(r\bar{a}) + \int_0^\infty e^{-\rho s}U_0'(r\bar{a})(c_{t+s} - r\bar{a})ds
\]

\[
\approx \frac{1}{\rho}U_0(r\bar{a}) + \int_0^\infty e^{-\rho s}U_0'(r\bar{a})\Psi(a_{t+s} - \bar{a})ds
\]

\[
\approx \frac{1}{\rho}U_0(r\bar{a}) + \int_0^\infty e^{-\rho s}U_0'(r\bar{a})\Psi(a_t - \bar{a})e^{(r-\Psi)s}ds
\]

\[
= \frac{1}{\rho}U_0(r\bar{a}) + U_0'(r\bar{a})\frac{\Psi}{\rho + \Psi - r}(a_t - \bar{a}).
\]

Therefore, \(W'(\bar{a}) = U_0'(r\bar{a})\frac{\Psi}{\rho + \Psi - r}\).

By differentiating \((8a)\), we obtain, at \(a^*\), \((\rho + \lambda - r)V'(\bar{a}) = \lambda W'(\bar{a})\). In addition \(V'(\bar{a}) = U_1'(\dot{c}(\bar{a}))U_1'(r\bar{a})\). Therefore

\[(\rho + \lambda - r)U_1'(r\bar{a}) = \lambda \frac{\Psi}{\rho + \Psi - r}U_0'(r\bar{a})\]

or equivalently

\[
\Psi = \frac{(r - \rho)(\rho + \lambda - r)}{\rho + \lambda - r - \lambda(\beta(r\bar{a}))} > 0,
\]

since \(\rho + \lambda - r > \lambda\beta(r\bar{a}) > 0\). We can verify that because \(\beta(r\bar{a}) > \bar{\beta}, \Psi > r\).

Proof of Proposition 4. First of all let \(\alpha_1 = \lim_{c \uparrow c^*} \beta(c)\) and \(\alpha_2 = \lim_{c \downarrow c^*} \beta(c)\). Assumption 4 implies that

\[
\bar{\beta} \leq \alpha_2 < \alpha_1 \leq 1.
\]

Let \(\bar{r} = \rho\frac{\rho + \lambda}{\rho + \lambda(1 - \alpha_2)}\) and \(\bar{r} = \rho\frac{\rho + \lambda}{\rho + \lambda(1 - \alpha_1)}\). From the inequalities above,

\[
\rho < \bar{r} < \bar{r}.
\]

For \(r \in (\bar{r}, \bar{r})\) we start at \(a^* = \frac{\bar{c}^*}{r}\) and \((V(a^*), W(a^*)) = (\bar{V}(a^*), \bar{W}(a^*))\). We show that the ODE \((28)\) admits a solution over \([a^*, \infty)\) with the initial condition at \(a^*\). Similarly, we
show that the ODE (29) admits a solution over \([a, a^*]\), with the initial condition at \(a^*\). Combining the two solutions, we obtain a Markov equilibrium defined over \([a, \infty)\).

Indeed, starting at \(a^* = \frac{\epsilon}{\tau}\), and the initial condition \((\overline{V}(a^*), \overline{W}(a^*))\), because \(r < \rho \frac{\rho + \lambda}{\rho + \lambda (1 - a_2)}\), \(\beta(ra^*_+) < \hat{\beta}\), we can use Lemma 1, to show the existence of a solution \((V, W)\) to ODE (28), given the initial condition. The solution has a (right) maximal interval of existence \([a^*, \tilde{a}]\). If \(1 - \alpha_2 > \sigma(U_1, c)\) for all \(c \geq c^*\), Theorem 8 shows that \(\tilde{a} = +\infty\), for \(\lambda\) sufficiently high. Otherwise, we follow the steps in the proof of Theorem 3 to restart the procedure each time \(V\) crosses \(\overline{V}\). In doing so, we obtain a Markov equilibrium over \([a^*, \infty)\) with \(\hat{\epsilon}(a) > ra\) except for a countable set of steady states at which \(\hat{\epsilon}(a) = ra\).

Starting at \(a^* = \frac{\epsilon}{\tau}\), and the initial condition \((\overline{V}(a^*), \overline{W}(a^*))\), we also show that we can extend the ODE (29) to the left until \(\tilde{a} < a^*\). First consider the case \(\alpha_1 < 1\) (the case \(\alpha_1 = 1\) will be considered below). The proof follows closely the steps of Lemma 1, i.e. we start with the initial condition

\[
\left(\overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \delta(\epsilon) \epsilon\right),
\]

where \(\delta(\epsilon) \in \left[1, \frac{\rho + \lambda}{\lambda}\right]\) is chosen appropriately. In Lemma 18 below, we show that there exists \(\epsilon > 0\) such that for all \(0 < \epsilon < \tilde{\epsilon}\), \(\delta(\epsilon)\) can be chosen such that

\[
\max \left\{ (\rho + \lambda - r) V_\epsilon'(a^*), \lambda U_\epsilon'(\hat{\epsilon}(a^*)) \right\} < \lambda W_\epsilon'(a^*) < \lambda V_\epsilon'(a^*).
\]

Therefore, following the steps in Lemma 12, we can show that

\[
\max \left\{ (\rho + \lambda - r) V_\epsilon'(a), 0 \right\} < \lambda W_\epsilon'(a) < \lambda V_\epsilon'(a),
\]

for all \(a\) in the (left) maximal interval of existence for \(V_\epsilon, W_\epsilon\).

As in the proof of Lemma 1, we show that there exists \(\epsilon > 0\) and \(\omega > 0\) such that the ODE (29) with the initial condition (56) admits a unique solution \((V_\epsilon, W_\epsilon)\) defined over \([a^* - \omega, a^*]\). Moreover, \(V_\epsilon(a) > \overline{V}(a)\) for \(a < a^*\).\(^{40}\) Therefore, we follow the steps in Lemma 12 to show that \(V_\epsilon''(a) < 0\), for all \(0 < \epsilon < \tilde{\epsilon}\) and \(a^* - \omega \leq a \leq a^*\).

Now let \(\bar{a} = a^* - \frac{\omega}{\tau}\), we have

\[
\overline{V}(a^*) + \epsilon - \sqrt{V(a^*) - \omega} \geq V_\epsilon(a^*) - V_\epsilon(a^* - \omega) \geq V_\epsilon(\bar{a}) - V_\epsilon(a^* - \omega) > \frac{\omega}{2} V_\epsilon'(\bar{a}),
\]

\(^{40}\)We also prove by contradiction: if \(V_{\epsilon_N}(a_N) = \overline{V}(a_N)\) and \(a_N \to a^*\) as \(N \to \infty\),

\[
\frac{V_{\epsilon_N}(a^*) - \overline{V}(a_N)}{a^* - a_N} \geq \frac{\overline{V}(a^*) - \overline{V}(a_N)}{a^* - a_N} = R_b(\tilde{a}_N, V_{\epsilon_N}(\tilde{a}_N), W_{\epsilon_N}(\tilde{a}_N))
\]

which at the limit contradicts the condition that \(\overline{V}'(a^-_\omega) > U_\epsilon'(ra^*_\omega)\) since \(\beta(ra^*_\omega) > \hat{\beta}\).
where the last inequality comes from the concavity of $V$. So $V'(a) < \frac{2}{\omega} (\overline{V}(a^*) + \bar{e} - \overline{V}(a^* - \omega))$

Also by the concavity of $V$

$$V'(a) \leq V'(a) < \frac{2}{\omega} (\overline{V}(a^*) + \bar{e} - \overline{V}(a^* - \omega)),$$

for all $a \in [a, a^*]$.

Together with (57), we have

$$0 < V'(a), W'(a) < \frac{2}{\omega} (\overline{V}(a^*) + \bar{e} - \overline{V}(a^* - \omega))$$

for all $a \in [a, a^*]$ and $\epsilon \in (0, \bar{e})$. Therefore, as in Lemma 1, we can apply Dominated Convergence Theorem to show that $(V, W)$ is a solution to the ODE (29) over $[a, a^*]$. Furthermore, for all $a \in (a, a^*], (a, V(a), W(a))$ is a regular point.

When $\beta(c) = 1 - a_1$ for $c < c^*$. Consider extending the solution $(V, W)$ to the ODE (29) from $a^*$ with the initial conditions $(V(a^*), W(a^*)) = (\overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \epsilon)$. As shown above, for $\epsilon$ sufficiently small, at $a^*$,

$$W''(a^*) < 0$$

$$W'(a^*) \geq U'(\hat{c}(a^*))$$

$$V(a^*) > \overline{V}(a^*).$$

By Lemmas 12, 15, and 16 (when $\lim_{c \uparrow c^*} \beta(c) = 1$, $W'(c) = V'(c) = U'(\hat{c}(c)) = U'(\hat{c}(c))$ for $a < a^*$, and $W'(c) > U'(\hat{c}(c))$ for $a < a^*$), these properties hold for all $a < a^*$ in the (left) maximal interval of existence of $(V, W)$ as a solution to ODE (29). We show by contradiction that $(V, W)$ can be extended all the way to $a = 0$. Assume that ODE (29) reaches a singular point at $\hat{a} > 0$. By Lemma 14, $V'(\hat{a}) = \overline{V}(\hat{a})$. Because $V(a) > \overline{V}(a)$ for all $a > \hat{a}$, $V'(\hat{a}) \geq \overline{V}'(\hat{a})$. Because $\hat{a}$ is a singular point, $V'(\hat{a}) = U'(r\hat{a})$. So $U'(r\hat{a}) \geq \overline{V}'(\hat{a})$, or equivalently $\beta(r\hat{a}) = 1 - a_1 \leq \hat{\beta}$, by Lemma 5. This contradicts the assumption that $r < \hat{r}$. So we obtain the desired contradiction. Thus, $(V, W)$ can be extended all the way to 0. Given $\omega > 0$ small

$$\overline{V}(a^*) + \epsilon - \overline{V}(a) \geq V(a^*) - V(a) > \overline{V}(a) - \overline{V}(a) > \omega V'(a),$$

where the last inequality comes from the concavity of $V$. So $V'(a) < \frac{2}{\omega} (\overline{V}(a^*) + \epsilon - \overline{V}(a^*)).$
Also by the concavity of $V_\epsilon$

$$0 < V_\epsilon'(a) \leq V_\epsilon'(\omega) < \frac{2}{\omega} \left( \overline{V}(a^*) + \epsilon - \overline{V}(\omega/2) \right),$$

for all $a \in [\omega, a^*]$. $W'_\epsilon$ are bounded by the same bounds. Similar to the proof of Lemma 1, there exists a subsequence $\epsilon \to 0$ such that $(V_\epsilon, W_\epsilon)$ converges to some limit $(V, W)$ which is a solution to ODE (29), with the initial condition $(a^*, \overline{V}(a^*), \overline{W}(a^*))$. By taking a sequence $\omega_n \to 0$, it is easy to show that the limit value functions $(V, W)$ can be extended until 0.

Proof. Because

$$\rho > \tau = \frac{\rho + \lambda - r}{\rho + \lambda(1 - a_2)} > \rho,$$

we have $(\rho + \lambda - r) < \lambda$. First, we show that there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and $\delta \in \left[1, \frac{\rho + \lambda}{\lambda}\right]$,

$$U'_0(\hat{c}_{\epsilon, \delta}(a^*)) < V'_{\epsilon, \delta}(a^*)$$

(58)

where

$$(V'_{\epsilon, \delta}(a^*), W'_{\epsilon, \delta}(a^*)) = (R_h, S_h) (a^*, \overline{V}(a^*) + \epsilon, \overline{W}(a^*) + \delta \epsilon)$$

and

$$\hat{c}_{\epsilon, \delta}(a^*) = (U'_1)^{-1} (V'_{\epsilon, \delta}(a^*)) < ra^*.$$

Because $R_h$ is continuous,

$$\lim_{\epsilon \to 0} V'_{\epsilon, \delta}(a^*) = U'_1(ra^*)$$

$$\lim_{\epsilon \to 0} \hat{c}_{\epsilon, \delta}(a^*) = ra^*.$$
\[ \frac{\rho + \lambda - r}{\lambda} < 1, \] this also implies
\[
\max \left\{ \frac{\rho + \lambda - r}{\lambda} V'_{\epsilon, \delta}(a^*), U_0'(\hat{\epsilon}_{e, \delta}(a^*)) \right\} < V'_{\epsilon, \delta}(a^*).
\]

By Lemma 19, we can choose \( \tilde{\epsilon} \) such that for all \( \epsilon \in (0, \tilde{\epsilon}) \), \( W'_{\epsilon, 1}(a^*) < V'_{\epsilon, 1}(a^*) \). It is easy to see that
\[
\lim_{\delta \uparrow \frac{\rho + \lambda}{\lambda}} W'_{\epsilon, \delta}(a^*) = +\infty > \max \left\{ \frac{\rho + \lambda - r}{\lambda} V'_{\epsilon, \delta}(a^*), U_0'(\hat{\epsilon}_{e, \delta}(a^*)) \right\}.
\]

So by the Intermediate Value Theorem, there exists \( \delta(\epsilon) \) such that
\[
\max \left\{ (\rho + \lambda - r) V'_{\epsilon}(a^*), \lambda U'_0(\hat{\epsilon}(a^*)) \right\} < \lambda W'_{\epsilon}(a^*) < \lambda V'_{\epsilon}(a^*).
\]

\[ \square \]

**Lemma 19.** For \( \epsilon > 0 \), let \( V_{\epsilon}(a^*) = V(a^*) + \epsilon \) and \( W_{\epsilon}(a^*) = W(a^*) + \epsilon \). We have
\[
\lim_{\epsilon \to 0} S_h(a^*, V_{\epsilon}(a^*), W_{\epsilon}(a^*)) = U'_0(ra^*).
\]

**Proof.** From the definition of \( R_h, S_h, \)
\[
V'_{\epsilon}(a^*) = R_h(a^*, V_{\epsilon}(a^*), W_{\epsilon}(a^*))
\]
\[
W'_{\epsilon}(a^*) = S_h(a^*, V_{\epsilon}(a^*), W_{\epsilon}(a^*)).
\]

Also by the definition of \( V_{\epsilon}(a^*), W_{\epsilon}(a^*) \), \( (\lambda + \rho) V_{\epsilon}(a^*) - \lambda W_{\epsilon}(a^*) = U_1(ra^*) + \rho \epsilon \). Using Taylor expansion for \( H(p, a) \) around \( p^* = U'_1(ra^*) \),
\[
\lambda \epsilon + U_1(ra^*) = H(V'_{\epsilon}(a^*), a^*)
\]
\[
= H(p^*, a^*) + \frac{\partial H(p^*, a^*)}{\partial p}(V'_{\epsilon}(a^*) - p^*)
\]
\[
+ \frac{1}{2} \frac{\partial^2 H(p^*, a^*)}{\partial p^2}(V'_{\epsilon}(a^*) - p^*)^2 + o((V'_{\epsilon}(a^*) - p^*)^2).
\]

From the proof of Lemma 3, \( H(p^*, a^*) = U_1(ra^*) \) and \( \frac{\partial H(p^*, a^*)}{\partial p} = 0 \). In addition,
\[
\frac{\partial^2 H(p^*, a^*)}{\partial p^2} = -\frac{1}{U''_1 \left( (U'_1)^{-1}(p^*) \right)} = -\frac{1}{U''_1(ra^*)} > 0.
\]

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\[ \rho \epsilon = -\frac{1}{2U_1''(ra^*)}(V'(a^*) - U'_1(ra^*))^2 + o((V'(a^*) - p^*)^2). \]

Consequently,

\[ V'(a^*) - U'_1(ra^*) = \sqrt{(-2U_1''(ra^*)) \rho \epsilon + o(\sqrt{\epsilon})}. \]

By the definition of \( \hat{c}_e \),

\[ \hat{c}_e(a^*) - ra^* = (U'_1)^{-1}(V'_e(a^*)) - (U'_1)^{-1}(U'_1(ra^*)) = \frac{1}{U_1''(ra^*)}(V'_e(a^*) - U'_1(ra^*)) + o(V'_e(a^*) - U'_1(ra^*)) = \frac{1}{U_1''(ra^*)} \sqrt{(-2U_1''(ra^*)) \rho \epsilon + o(\sqrt{\epsilon})}. \]

Therefore,

\[ W'(ra) = \frac{\rho W_e(a^*) - U_0(\hat{c}_e(a^*))}{ra^* - \hat{c}_e(a^*)} = \frac{U_0(ra^*) - U_0(\hat{c}_e(a^*)) - \rho \epsilon}{ra^* - \hat{c}_e(a^*)} = \frac{\rho \epsilon}{-U_1''(ra^*) \sqrt{(-2U_1''(ra^*)) \rho \epsilon + o(\sqrt{\epsilon})}} \rightarrow U'_0(ra^*), \]

as \( \epsilon \rightarrow 0 \).

\[ \square \]

I  Proofs for Further Characterizations

**Proof of Theorem 7.** Because \( r < \rho \), there exists \( \epsilon \in (0, 1) \) (sufficiently small) such that

\[ \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} > 1 \]

and

\[ (\rho - r) \frac{1 - \epsilon}{\lambda} \frac{1 - \epsilon}{\epsilon} > \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda}. \]

In the proof of Lemma 1, we show that

\[ \lim_{a \downarrow a} W'(a) \leq U'_0(ra) \leq U'_1(ra) = \lim_{a \downarrow a} V'(a). \]
Therefore, in the right neighborhood of \( a \),

\[
\lambda W'(a) < (1 - \epsilon) (\lambda + \rho - r) V'(a),
\]

because \((1 - \epsilon) (\lambda + \rho - r) > \lambda\).

We use Lemma 6 to show that \( \lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r)V'(a) \) for all \( a > \tilde{a} \). We just showed that this is true in the right neighborhood of \( a \), so the first condition in Lemma 6 is satisfied. Now, we show that the second (relax) condition in Lemma 6 is also satisfied, i.e. if there exists \( \tilde{a} > a \) such that \( \lambda W'(\tilde{a}) = (1 - \epsilon)(\rho + \lambda - r)V'(\tilde{a}) \), then \( \lambda W''(a) < (1 - \epsilon)(\rho + \lambda - r)V''(a) \) in the left neighborhood of \( \tilde{a} \).

Indeed, in the left neighborhood of \( \tilde{a} \), \( \lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a) \), therefore

\[
V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon(\rho + \lambda - r)V'(a)}{\hat{c}(a) - ra} < 0,
\]

Differentiating equation (8b)

\[
W''(a) = \frac{(U_0'(\hat{c}(a))\hat{c}'(a) - \rho W'(a)) (\hat{c}(a) - ra) - (U_0(\hat{c}(a)) - \rho W(a))(\hat{c}'(a) - r)}{\hat{c}(a) - ra} - \frac{(\hat{c}(a) - ra)^2}{(\hat{c}'(a) - r)}.
\]

Therefore,

\[
W''(a) = \frac{(U_0'(\hat{c}(a)) - W'(a)) \hat{c}'(a)}{\hat{c}(a) - ra} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra} = \frac{(U_0'(\hat{c}(a)) - W'(a))}{\hat{c}(a) - ra} \frac{V''(a)}{U_1''(\hat{c}(a))} + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra} = \frac{(W'(a) - U_0'(\hat{c}(a))}{(\hat{c}(a) - ra) (-U_1''(\hat{c}(a)))} V''(a) + \frac{(r - \rho)W'(a)}{\hat{c}(a) - ra}.
\]

When \( a \) close to \( \tilde{a} \), we also have:

\[
V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\lambda}{1 - \epsilon} \frac{W'(a)}{\hat{c}(a) - ra},
\]

because, by continuity, when \( a \) close to \( \tilde{a} \), \( \lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a) \). Therefore, \( W'(a) \approx -\frac{1}{\lambda} \frac{1}{1 - \epsilon} V''(a)(\hat{c}(a) - ra) \). Plugging this back to the expression for \( W'' \) above, we
Let $F$ and $\dot{\beta}$ in the previous lemma, the last inequality comes from $V''(a) < 0$. Therefore both conditions 1) and 2) in Lemma 6 are satisfied, and by that lemma, $\lambda W'(a) \leq (1 - \epsilon)(\rho + \lambda - r)V'(a)$ for all $a \leq a^*$.

We prove by contradiction that $a^*$ is infinite. Assume by contradiction that $a^*$ is finite. Let $F(a) = (\rho + \lambda)V(a) - \lambda W(a) - U_1(ra)$. At $a = a^*$, $F(a) = 0$ and

$$F'(a) = (\rho + \lambda)V'(a) - \lambda W'(a) - rU_1'(ra)$$

$$> (\rho + \lambda)V'(a) - (1 - \epsilon)(\rho + \lambda - r)V'(a) - rU_1'(ra)$$

$$= (\rho + \lambda - (1 - \epsilon)(\rho + \lambda - r) - r)U_1'(ra)$$

$$= \epsilon (\rho + \lambda - r) U_1'(ra) > 0.$$

So $F(a) < 0$ in the left neighborhood of $a^*$. This is a contradiction. Thus $a^* = +\infty$.

By Lemma 4,

$$\dot{\beta}'(a) = \frac{V''(a)}{U_1''(\dot{\beta}(a))} = \frac{(\lambda + \rho - r) V'(a) - \lambda W'(a)}{U_1''(\dot{\beta}(a))(\dot{\beta}(a) - ra)}$$

$$> \frac{\epsilon (\lambda + \rho - r)V'(a)}{U_1''(\dot{\beta}(a))(\dot{\beta}(a) - ra)} > \frac{\epsilon \lambda V'(a)}{U_1''(\dot{\beta}(a))(\dot{\beta}(a) - ra)} > 0,$$

where the last inequality comes from $r < \rho$. 

\begin{proof}[Proof of Theorem 8] Condition (13) implies that $\sup_{c > 0} \beta(c) < 1$. Therefore, there exists

\end{proof}
\(\epsilon \in (0, 1)\) such that 
\[
\beta(c) < 1 - \epsilon
\]
and
\[
(1 - \epsilon) - \beta(c) > (1 - \epsilon) \sigma(U_1, c)
\]
for all \(c > 0\).\(^{41}\) Therefore, given \(\rho\) and \(\lambda\), there exists \(\rho + \lambda > \bar{r} > \rho\) such that for all \(r \in [\rho, \bar{r}]\), we have:
\[
\lambda U'_0(ra) < (1 - \epsilon)(\lambda + \rho - r) U'_1(ra) \tag{59}
\]
and for all \(c > 0\),
\[
\frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} - \frac{U'_0(c)}{U'_1(c)} > \left(\frac{r - \rho}{\lambda} - \frac{1 - \epsilon}{\epsilon} + \frac{1 - \epsilon}{\lambda} \right) \sigma(U_1, c). \tag{60}
\]
Moreover \(\bar{r}\) can be chosen to be increasing in \(\lambda\) and \(\lim_{\lambda \to \infty} \bar{r}(\lambda) = \infty\).

By the definition of \(\bar{r}\), for all \(r \in [\rho, \bar{r}]\),
\[
\rho + \lambda - r > 0. \tag{61}
\]

In the proof of Lemma 1, we show that
\[
\lim_{a \downarrow \bar{a}} W'(a) \leq U'_0(ra), \\
\lim_{a \downarrow \bar{a}} V'(a) = (1 - \epsilon)U'_1(ra).
\]
Therefore, by (59),
\[
\lambda W'(a) < (1 - \epsilon)(\lambda + \rho - r) V'(a)
\]
in the right neighborhood of \(\bar{a}\).

Given these three conditions, as in the proof of Theorem 7, we use Lemma 6 to show that \(\lambda W'(a) < (1 - \epsilon)(\rho + \lambda - r)V'(a)\) for all \(a > \bar{a}\). As shown above, this is true in the right neighborhood of \(\bar{a}\) so the first condition in Lemma 6 is satisfied. Now we show that the second (relax) condition in Lemma 6 is also satisfied, i.e. if there exists \(\bar{a} > \bar{a}\) such that \(\lambda W'(\bar{a}) = (1 - \epsilon)(\rho + \lambda - r)V'(\bar{a})\), we show that \(\lambda W''(a) < (1 - \epsilon)(\rho + \lambda - r)V''(a)\) in the left neighborhood of \(\bar{a}\). Indeed, in the left neighborhood of \(\bar{a}\), \(\lambda W'(a) \approx (1 - \epsilon)(\rho + \)

\(^{41}\)This is equivalent to \(1 - \sigma(U_1, c) > \frac{1}{1 - \epsilon} \beta(c)\), which is true given (13).
\[ \lambda - r) V'(a), \text{ therefore} \]

\[ V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\epsilon (\rho + \lambda - r) V'(a)}{\hat{c}(a) - ra} < 0, \]

Differentiating equation (8b) and simplifying as done in the proof of Theorem 7:

\[ W''(a) = \frac{(W'(a) - U'_0(\hat{c}(a)))}{(\hat{c}(a) - ra) (-U''_1(\hat{c}(a)))} V''(a) + \frac{(r - \rho) W'(a)}{\hat{c}(a) - ra}. \]

When \( a \) close to \( \hat{a} \), we also have:

\[ V''(a) = \frac{\lambda W'(a) - (\lambda + \rho - r) V'(a)}{\hat{c}(a) - ra} \approx -\frac{\lambda \epsilon W'(a)}{\hat{c}(a) - ra}, \]

because, by continuity, when \( a \) close to \( \hat{a} \), \( \lambda W'(a) \approx (1 - \epsilon)(\rho + \lambda - r)V'(a) \). Therefore, \( W'(a) \approx -\frac{1 - \epsilon}{\lambda} V''(a)(\hat{c}(a) - ra) \). Plugging this back to the expression for \( W'' \) above, we have

\[ W''(a) \approx \left( \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} U'_1(\hat{c}(a)) - \frac{U'_0(\hat{c}(a))}{\hat{c}(a) - ra} \right) V''(a) - \frac{(r - \rho) \frac{1 - \epsilon}{\lambda} V''(a)(\hat{c}(a) - ra)}{\hat{c}(a) - ra} \]

\[ = \left( \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} U'_1(\hat{c}(a)) - \frac{U'_0(\hat{c}(a))}{\hat{c}(a) - ra} \right) \frac{1}{\sigma(U_1, \hat{c}(a))} \frac{\hat{c}(a)}{\hat{c}(a) - ra} - \frac{(r - \rho) \frac{1 - \epsilon}{\lambda}}{\hat{c}(a) - ra} \right) V''(a) \]

\[ < \left( \frac{(1 - \epsilon)(\rho + \lambda - r)}{\lambda} V''(a) \right), \]

where the last inequality comes from (60) and \( V''(a) < 0 \). Therefore both conditions 1) and 2) in Lemma 6 are satisfied, and by that lemma, \( \lambda W'(a) \leq (1 - \epsilon)(\rho + \lambda - r)V'(a) \) for all \( a \leq a^* \).

As in the proof of Theorem 7, we prove by contradiction that \( a^* \) is infinite. Assume by contradiction that \( a^* \) is finite. Let \( F(a) = (\rho + \lambda)V(a) - \lambda W(a) - U_1(ra) \). At \( a = a^* \), \( F(a) = 0 \) and

\[ F'(a) = (\rho + \lambda)V'(a) - \lambda W'(a) - rU'_1(ra) \]

\[ > (\rho + \lambda)V'(a) - (1 - \epsilon)(\rho + \lambda - r)V'(a) - rU'_1(ra) \]

\[ = (\rho + \lambda - (1 - \epsilon)(\rho + \lambda - r) - r)U'_1(ra) \]

\[ = \epsilon (\rho + \lambda - r) U'_1(ra) > 0, \]

where the last inequality comes from (61). So \( F(a) < 0 \) in the left neighborhood of \( a^* \). This
is a contradiction. Thus $a^* = +\infty$.

Similar to the proof of Theorem 7,

$$
\hat{c}'(a) = \frac{(\lambda + \rho - r) V'(a) - \lambda W'(a)}{U_1'(\hat{c}(a)) (\hat{c}(a) - ra)} > \frac{\lambda(\lambda + \rho - r) V'(a)}{U_1'(\hat{c}(a)) (\hat{c}(a) - ra)} > 0,
$$

where the last inequality also comes from (61).

**Proof of Theorem 9.** We use the notation $V_\lambda, W_\lambda$ for the functions defined in (10). Notice that $W_\lambda(a) = \frac{1}{\rho} U_0(ra)$, independent of $\lambda$, so we can drop the subscript $\lambda$. We prove this theorem by contradiction. Assume that the result does not hold, then there exists $\bar{a} > a$ and a sequence of $\{\lambda_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} \lambda_n = \infty$, such that $V_{\lambda_n}(a) > \bar{V}_{\lambda_n}(a)$ for all $a \in (a, \bar{a})$.\footnote{Because $\lim_{a \downarrow \bar{a}} \frac{U_0(ra) - U_0(ra)}{a - \bar{a}} = rU_0'(a)$, and $r > \rho$, there exists $a_1 \in (a, \bar{a})$ and $0 < \gamma$ such that $\frac{1}{\rho} U_0(ra_1) - U_0(ra) > (\gamma + 1) U_0'(ra)$.}

First, using Lemma 20 below, we show that $\lim_{n \to \infty} W_{\lambda_n}(a_1) = W(a_1)$.

Indeed, by Lemma 10, $W_{\lambda_n}(a_1) - \bar{W}(a_1) \leq 0$. Therefore

$$
\limsup_{n \to \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) \leq 0. \tag{62}
$$

Now,

$$
W_{\lambda_n}(a_1) - \bar{W}(a_1) = W_{\lambda_n}(a_1) - V_{\lambda_n}(a_1) + V_{\lambda_n}(a_1) - \bar{V}_{\lambda_n}(a_1)
+ \bar{V}_{\lambda_n}(a_1) - \bar{W}_{\lambda_n}(a_1).
$$

By Lemma 20,

$$
\lim_{n \to \infty} (V_{\lambda_n}(a_1) - W_{\lambda_n}(a_1)) = 0.
$$
By the definition of $\bar{V}_\lambda, \bar{W}_\lambda$ in (10)

$$\lim_{n \to \infty} (\bar{V}_{\lambda_n}(a) - \bar{W}_{\lambda_n}(a)) = 0,$$

and by the contradiction assumption

$$V_{\lambda_n}(a_1) - \bar{V}_{\lambda_n}(a_1) \geq 0.$$

Thus

$$\liminf_{n \to \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) \geq 0. \quad (63)$$

Therefore by (62) and (63)

$$\lim_{n \to \infty} (W_{\lambda_n}(a_1) - \bar{W}(a_1)) = 0.$$

Given this limit, for $\epsilon > 0$, sufficiently small, there exists $N$ such that $W_{\lambda_n}(a_1) - \bar{W}(a_1) > -\epsilon$ for all $n \geq N$. Now,

$$\frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(a)}{a_1 - a} > \frac{\bar{W}(a_1) - \bar{W}(a)}{a_1 - a} > (\gamma + 1) U'_0(ra) - \frac{\epsilon}{a_1 - a}.$$

By the Mean Value Theorem, there exists $a_n \in (a, a_1)$ such that,

$$W_{\lambda_n}'(a_n) = \frac{W_{\lambda_n}(a_1) - W_{\lambda_n}(a)}{a_1 - a} \leq U'_0(ra_n) + \frac{\rho}{\lambda_n} U'_1(ra_n) \leq U'_0(ra) + \frac{\rho}{\lambda_n} U'_1(ra),$$

where the first inequality comes from the proof of Lemma 9 (especially inequality (41)). By choosing $\epsilon$ sufficiently small and $n$ sufficiently large such that

$$\frac{\epsilon}{a_1 - a} + \frac{\rho}{\lambda_n} U'_1(ra) < \gamma U'_0(ra),$$

we obtain the desired contradiction.

Lemma 20. Assume that there exists $\bar{a} > a$ and a diverging sequence $\{\lambda_n\}$ such that $V_{\lambda_n}(a) > \bar{V}_{\lambda_n}(a)$ for all $a \in (a, \bar{a})$. Then

$$\lim_{n \to \infty} (V_{\lambda_n}(a) - W_{\lambda_n}(a)) = 0, \quad (64)$$
for all \( a \in (a, \bar{a}) \).

**Proof.** By Lemma 10, \( W_{\lambda_n} \leq \bar{W} \) therefore

\[
V_{\lambda_n}(a) - W_{\lambda_n}(a) \geq V_{\lambda_m}(a) - W_{\lambda_n}(a)
\]

for all \( a \in (a, \bar{a}) \).

To find an upper bound on \( V_{\lambda_n} - W_{\lambda_n} \). We rewrite equation (8a) as

\[
\lambda (V_{\lambda}(a) - W_{\lambda}(a)) = U_1(\hat{c}_\lambda(a)) + V_{\lambda}'(a)(ra - \hat{c}_\lambda(a)) - \rho V_{\lambda}(a).
\]

Therefore

\[
\lambda (V_{\lambda}(a) - W_{\lambda}(a)) \leq U_1(\hat{c}_\lambda(a)) - \rho V_{\lambda}(a), \tag{65}
\]

because \( V_{\lambda}(a) > V_{\lambda}(a) = V_{\lambda}(a) \), and \( V_{\lambda}' \geq 0 \), and \( ra - \hat{c}_\lambda(a) < 0 \).

Now if \( U_1 \) is bounded above

\[
\lambda (V_{\lambda}(a) - W_{\lambda}(a)) \leq \sup_{c} U_1(c) - \rho V_{\lambda}(a).
\]

Thus \( \lambda |V_{\lambda}(a) - W_{\lambda}(a)| \) is bounded when \( \lambda \to \infty \). Therefore (64) holds.

If \( U_1 \) is not bounded, but Condition (14) is satisfied, we show, using Lemma 6, that there exists \( \bar{\lambda} \) such that, when \( \lambda > \bar{\lambda} \), \( \hat{c}_\lambda(a) < \frac{2\lambda}{\rho} a \), for all \( a \in (a, \bar{a}) \). Let \( f(a) = \frac{2\lambda}{\rho} a \) and \( g(a) = \hat{c}_\lambda(a) \). With \( \lambda > \sigma r \), \( f(a) = \frac{2\lambda}{\rho} > ra \). We just need to verify that if \( f(a) = g(a) \) then \( f'(a) = \frac{2\lambda}{\rho} > g'(a) = \hat{c}_\lambda'(a) \). Indeed, by differentiating, the first order condition (11) with respect to \( a \),

\[
\hat{c}_\lambda'(a) = \frac{V_{\lambda}''(a)}{U_1''(\hat{c}_\lambda(a))}.
\]

To get \( V_{\lambda}''(a) \), differentiating (8a) with respect to \( a \) and use the first order condition for \( c \):

\[
V_{\lambda}''(a)(ra - \hat{c}_\lambda(a)) = (\rho + \lambda - r) V_{\lambda}'(a) - \lambda W_{\lambda}'(a) \\
= (\rho + \lambda - r) U_1'(\hat{c}_\lambda(a)) - \lambda W_{\lambda}'(a).
\]
Therefore, because \( W_\lambda' \geq 0 \) as shown in Lemma 9,

\[
\hat{c}_\lambda'(a) = \frac{(\rho + \lambda - r) U_1'(\hat{c}_\lambda(a)) - \lambda W_\lambda'(a)}{-U_1''(\hat{c}_\lambda(a)) (\hat{c}_\lambda(a) - ra)} \leq \frac{(\rho + \lambda - r) U_1'(\hat{c}_\lambda(a))}{-U_1''(\hat{c}_\lambda(a)) (\hat{c}_\lambda(a) - ra)}.
\]

\[
= (\rho + \lambda - r) \frac{1}{\sigma(U_1, \hat{c}_\lambda(a))} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)} < \frac{\lambda}{\sigma} \frac{\hat{c}_\lambda(a)}{(\hat{c}_\lambda(a) - ra)}
\]

\[
= \frac{\lambda}{\sigma} \frac{2\lambda}{2\lambda - ra}.
\]

By choosing \( \lambda \) sufficiently large, for all \( \lambda > \bar{\lambda}, \frac{2\lambda}{2\lambda - ra} < 2 \) for all \( a \in (a, \bar{a}) \). Therefore, by Lemma 6, \( \hat{c}_\lambda(a) < \frac{2\lambda}{\sigma} a \).

Now, going back to inequality (65),

\[
\lambda (V_\lambda(a) - W_\lambda(a)) \leq U_1(\hat{c}_\lambda(a) - \rho \bar{V}_\lambda(a)) < U_1 \left( \frac{2\lambda}{\sigma} \bar{a} \right) - \rho \bar{V}_\lambda(a).
\]

By the INADA conditions

\[
\lim_{\lambda \to \infty} \frac{U_1 \left( \frac{2\lambda}{\sigma} \bar{a} \right)}{\lambda} = 0.
\]

It is easy to show that \( \lim_{\lambda \to \infty} \frac{V_\lambda(a)}{\lambda} = 0 \). Thus we obtain the desired convergence (64).

**Proof of Theorem 10.** Given \( r > \rho \). First, by Theorem 9, for any \( a > a \), there exists \( \bar{\lambda}_1 \) such that for any \( \lambda > \bar{\lambda}_1 \), \( V(a) \) as constructed in Theorem 3 crosses \( \bar{V}(a) \) at some \( \bar{a} \in (a, a) \). Therefore, following the construction in Theorem 3, there exists a Markov equilibrium with multiple steady states. Second, applying Theorem 8, there exists \( \bar{\lambda}_2 \) such that for all \( \lambda > \bar{\lambda}_2, (\bar{r}(\lambda) > r \text{ for all } \lambda \geq \bar{\lambda}_2) \) there exists a continuous Markov equilibrium. Thus for \( \lambda \geq \bar{\lambda} = \max \{ \bar{\lambda}_1, \bar{\lambda}_2 \} \), there exist at least two Markov equilibria. \[43\]

**J Proofs for Instantaneous Gratification Limit**

**Proof of Proposition 5.** We re-write the system (15) as

\[
\rho W(a) = H_\infty(a, W'(a)) \tag{66}
\]

\[43\]Furthermore, we can apply Theorem 8 to construct a continuous Markov equilibrium starting from any \( a_n \), defined in Theorem 3. Doing so we obtain more than two Markov equilibria.
where
\[ H_\infty(a, p) = U_0 \left( (U'_1)^{-1}(p) \right) + p \left( ra - (U'_1)^{-1}(p) \right). \]

Now,
\[
\frac{\partial H_\infty(a, p)}{\partial p} = \left( U'_0 \left( (U'_1)^{-1}(p) \right) - p \right) \frac{1}{U''_1 \left( (U'_1)^{-1}(p) \right)} + \left( ra - (U'_1)^{-1}(p) \right)
\geq \left( U'_1 \left( (U'_1)^{-1}(p) \right) - p \right) \frac{1}{U''_1 \left( (U'_1)^{-1}(p) \right)} + \left( ra - (U'_1)^{-1}(p) \right)
= \left( ra - (U'_1)^{-1}(p) \right).
\]

Therefore \( \frac{\partial H_\infty(a, p)}{\partial p} > 0 \) if \( ra > (U'_1)^{-1}(p) \). Moreover, \( H_\infty(a, U'_1(ra)) = U_0(ra) \), and \( \lim_{p \to \infty} \frac{\partial H(a, p)}{\partial p} = +\infty \), as shown in Lemma 3. These properties allow us to conclude that if \( W(a) > \frac{1}{\rho} U_0(ra) \) there exists an unique solution \( W'(a) \) of (66) with \( \hat{c}(a) = (U'_1)^{-1}(W'(a)) < ra. \)

To determine the value functions at high assets, \( a \geq a_u \), we treat the two cases in Assumption 2 separately. In either case, we show that at \( a = a_u \), \( W(a) > \frac{1}{\rho} U_0(ra) \). Thus starting from \( a = a_u \), we can extend the \( W \) to the left following the ODE (66), with \( \hat{c}(a) < ra. \)

Case 1: \( \beta(c) = 1 \) for all \( c \geq \bar{c} \). By the condition stated in the proposition, \( \beta(\bar{c}) = 1 > \frac{\rho}{r} \), or equivalently \( r > \rho. \)

As shown Subsection A.3.2, \( W(a) = V_0^*(a) \) for all \( a \geq a_u \). Therefore,
\[
W(a_u) = V_0^*(a_u) > \frac{1}{\rho} U_0(ra_u).
\]

Case 2: \( \beta(c) = \tilde{\beta} > \frac{\rho}{r} \) and \( U_1(c) = \frac{1-\sigma}{1-\alpha} \) for all \( c \geq \bar{c} \).

Because \( \rho - r(1-\sigma) > 0 \)
\[
\tilde{\beta} + \sigma > \frac{\rho + \sigma r}{r} > 1.
\]

Direct algebras show that
\[
W(a) = \left( \frac{\Delta}{\sigma + \tilde{\beta} - 1} \right)^{-\sigma} \frac{a^{1-\sigma}}{1-\bar{c}}
\]
and
\[
\hat{c}(a) = \frac{\Delta}{\sigma + \tilde{\beta} - 1} a < ra
\]
satisfies (66) for all $a \geq a_u = \frac{\sigma + \beta - 1}{\Delta} \cdot 44$

It is easy to show that,

$$W(a_u) = \left(\frac{\Delta}{\sigma + \beta - 1}\right)^{-\sigma} \frac{a_u^{1-\sigma}}{1-\sigma} > \frac{1}{\rho} U_0(ra_u) = \frac{\tilde{\rho}}{\rho} (ra_u)^{1-\sigma}.$$ 

Now in either Case 1 or Case 2, starting from $a_u$, we can extend the $W$ to the left following the ODE (66), with $\hat{c}(a) < ra$. When we reach a singular point $\hat{a} < a_u$, we must have $W(\hat{a}) = \frac{1}{\rho} U_0(r\hat{a})$.

Between $\hat{a}$ and $a_u$, differentiating (66) with respect to $a$, we obtain:

$$(\rho - r) W'(a) = W''(a) \left( ra - \hat{c}(a) + (U_0'(\hat{c}(a)) - U_1'(\hat{c}(a))) \frac{1}{U_1'(\hat{c}(a))} \right).$$

Because $\rho < r$, $W'(a) > 0$, $ra > \hat{c}(a)$, $U_0' < U_1'$ and $U_1'' < 0$, we have $W''(a) < 0$. Therefore $\hat{c}'(a) = \frac{W''(a)}{U_1'(\hat{c}(a))} > 0$.

Now we turn to the last result of this proposition. Under Assumption 2a, we show that there exists $r_1 > \rho$ such that for all $r \in (\rho, r_1]$, $\hat{a} > 0$. Indeed, as in the proof of Proposition 3, for any $\bar{a} < \frac{\hat{c}}{\rho}$, there exists $r_1 > \rho$ such that for all $r \in (\rho, r_1]$,

$$W(a_u) = V_0(a_u) - \frac{\bar{a}}{\rho} < \int_{\bar{a}}^{a_u} U_1'(ra)da + \frac{1}{\rho} U_0(r\bar{a}). \quad (67)$$

We show by contradiction that $\hat{a} > \bar{a}$ for all $r \in (\rho, r_1]$. Assume by contradiction that $\hat{a} \leq \bar{a}$. Then

$$\frac{1}{\rho} U_0(r\hat{a}) < W(\hat{a}) = W(a_u) - \int_{\bar{a}}^{a_u} W'(a)da$$

$$< W(a_u) - \int_{\bar{a}}^{a_u} U_1'(ra)da,$$

because $W'(a) = U_1'(\hat{c}(a)) > U_1'(ra)$. This contradicts (67). So $\hat{a} > \bar{a}$. \qed

**Proof of Proposition 6.** In order to show this result, we work with $\hat{c}(a)$ instead of $W(a)$. Indeed, differentiate (15) with respect to $a$, we get

$$\rho W'(a) = U_0(\hat{c}(a))\hat{c}'(a) + W''(a)(ra - \hat{c}(a)) + W'(a)(r - \hat{c}'(a)). \quad (68)$$

Moreover, using the fact that $U_1'(\hat{c}(a)) = W'(a)$ and $U_1''(\hat{c}(a))\hat{c}'(a) = W''(a)$, the equation

\[44\text{It is easy to show that this is also the limit of the linear equilibria in Theorem 1 when } \lambda \to \infty.\]
above can be written as an equation for \( \hat{c} \):

\[
(\rho - r) U'_1(\hat{c}(a)) = (U_0(\hat{c}(a)) - U'_1(\hat{c}(a))) \hat{c}'(a) + U''_1(\hat{c}(a)) \hat{c}'(a) (ra - \hat{c}(a)).
\]

Equivalently,

\[
\hat{c}'(a) = \frac{(\rho - r) U'_1(\hat{c}(a))}{U_0(\hat{c}(a)) - U'_1(\hat{c}(a)) - U''_1(\hat{c}(a))(\hat{c}(a) - ra)} = \frac{\rho - r}{\frac{1}{\hat{c}(a)} - \frac{1 - \beta(\hat{c}(a))}{\sigma(U_1, \hat{c}(a))}}.
\]

(69)

Consider the solution to this ODE, with the initial condition at \( a, \hat{c}(a) = c \). By the definition of \( \hat{c} \)

\[
\frac{\hat{c} - ra}{c} > \frac{1 - \beta(ra)}{\sigma(U_1, ra)} > \frac{1 - \beta\left(\frac{\hat{c}}{c}\right)}{\sigma(U_1, \hat{c})},
\]

where the last inequality is due to the assumption \( \frac{1 - \beta(c)}{\sigma(U_1, c)} \) is weakly decreasing in \( c \) and \( \hat{c} > ra \). Therefore, the ODE (69) is well-defined and is regular at \( a \). Consequently, we can use the standard extensions theorem, to extend the solution to this ODE to the right. We show that, we can extend this solution to the whole interval \([a, \infty)\).

Let \( c^*(a) > 0 \) be determined by the solution of the equation

\[
\frac{c - ra}{c} = \frac{1 - \beta(c)}{\sigma(U_1, c)}.
\]

Because the left-hand side is strictly increasing in \( c \) and the right-hand side is weakly decreasing in \( c \). Moreover at \( c = ra \) the left-hand side being equal to 0 is strictly less than the right-hand side, which is greater than \( \epsilon > 0 \). At \( c \to \infty \), the left-hand side converges to 1 while the right-hand side is bounded by \( 1 - \epsilon \). So \( c^*(a) > ra \) exists and is unique.

Using Lemma 6, we show that \( \hat{c}(a) > c^*(a) \) for all \( a \geq a \). At \( a = a \), because \( \frac{\hat{c} - ra}{c} > 1 - \beta\left(\frac{\hat{c}}{c}\right) \), \( \hat{c}(a) = \hat{c} > c^*(a) \). We show that if \( \hat{c}(\tilde{a}) = c^*(\tilde{a}) \), at some \( \tilde{a} > a \), then in the left neighborhood of \( \tilde{a} \), \( \hat{c}'(a) > c^*(a) \). Indeed, because \( \hat{c}(\tilde{a}) = c^*(\tilde{a}) \), and \( (\rho - r)\sigma(U_1, \hat{c}(a)) > (\rho - r)\frac{1}{\sigma} > 0 \), by (69), \( \lim_{a \to \tilde{a}} \hat{c}'(a) = +\infty \), while \( c^*(a) \) is finite around \( \tilde{a} \). Therefore, \( \hat{c}'(a) > c^*(a) \) in the left neighborhood of \( \tilde{a} \). So both conditions in Lemma 6 are satisfied. Thus \( \hat{c}(a) > c^*(a) \) for all \( a > a \), so we can use standard extension theorem to extend \( \hat{c} \) to the whole interval \([a, \infty)\). Moreover, \( \hat{c}'(a) > 0 \) for all \( a \geq a \).

Given the solution \( \hat{c}(a) \), let \( W(a) = \frac{1}{\rho} U_0(ra) + \int_a^\infty U'_1(\hat{c}(\tilde{a}))d\tilde{a} \). Then \( W(a) = \frac{1}{\rho} U_0(ra) \)
and $W'(a) = U'_1(\hat{c}(a))$. We show that $W$ satisfies (15). Indeed, because of (68),

$$
\rho W(a) - U_0(\hat{c}(a)) - W'(a)(ra - \hat{c}(a))
= \rho W(a) - U_0(\hat{c}(a)) - W'(a)(ra - \hat{c}(a)).
$$

Moreover,

$$
\rho W(a) - U_0(\hat{c}(a)) - W'(a)(ra - \hat{c}(a))
= U_0(ra) - U_0(\bar{c}) - U'_1(\bar{c})(ra - \bar{c})
= 0,
$$

by the last condition in the statement of this proposition. Therefore, (15) is satisfied.

When $\sigma(U_1,c) \equiv \bar{\sigma}$, and $\beta(c) \equiv \bar{\beta}$ we show that there exists $\bar{c} > \frac{ra}{1-\bar{\sigma}}$ such that $F(\bar{c}) = 0$, where

$$
F(c) \equiv U_0(c) - U_0(ra) - U'_1(c)(c - ra).
$$

Because $\lim_{c \to ra} \frac{U_0(c) - U_0(ra)}{c - ra} = U'_0(ra) < U'_1(ra)$, $F(c) < 0$ in the right neighborhood of $ra$. Direct calculation shows that because $\bar{\sigma} + \bar{\beta} > 1$, $\lim_{c \to \infty} F(c) > 0$. By Intermediate Value Theorem, there exists $\bar{c} > ra$ such that $F(\bar{c}) = 0$.

Consider the first derivative of $F$:

$$
F'(c) = U'_0(c) - U'_1(c) - U''_1(c)(c - ra)
= U'_1(c)\sigma(U_1,c) \left( \frac{c - ra}{c} - \frac{1}{\bar{\sigma}} \right).
$$

Therefore $F'(c) \leq 0$ for $c \leq c^*(a)$. Consequently, $F(c) < 0$ for $c \leq c^*(a)$. Thus $\bar{c} > c^*(a) = \frac{ra}{1-\bar{\sigma}}$.

As shown above $\hat{c}'(a) > 0$, consider the inverse function of $\hat{c}$: $f(\hat{c}(a)) = a$. First of all, $f(\bar{c}) = a$. Second,

$$
1 = \hat{c}'(a) f'(\hat{c}(a)).
$$

Therefore, combining with (69) yields

$$
f'(c) = \frac{1}{\hat{c}'(a)} = \frac{1 - \frac{rf(c)}{c}}{(\rho - r)} \frac{1 - \bar{\beta}}{\bar{\sigma}},
$$

or equivalently,

$$
\frac{1}{r} \left( 1 - \frac{1 - \bar{\beta}}{\bar{\sigma}} \right) c = \left( \frac{\rho}{r} - 1 \right) \frac{1}{\bar{\sigma}} cf'(c) + f(c).
$$
So
\[
\left( c^{(\ell-1)\frac{1}{\rho}} f(c) \right)' = c^{(\ell-1)\frac{1}{\rho}} \frac{1}{r} \left( 1 - \frac{1}{\bar{\sigma}} \right).
\]
This ODE admits a unique solution in closed form:
\[
f(c) = c^{\frac{\bar{\beta} + \bar{\sigma} - 1}{\Delta}} + A_0 c^{-\frac{\rho - r}{\bar{\sigma}}},
\]
where \( A_0 \) is determined by \( f(\bar{c}) = a \).

\[\square\]

K Proofs for IG Equilibria with Uncertainty

Lemma 21. There exists \( \hat{U}(c) \) strictly increasing and strictly concave such that, for all \( p > 0 \),
\[
\max_{c > 0} \left\{ \hat{U}(c) - pc \right\} = U_0(\hat{c}) - p\hat{c},
\]
where \( \hat{c} = \arg \max_c U_1(c) - pc \).

Proof. Let \( \hat{U}(c) \) be determined such that
\[
c = \frac{U_1'(\hat{c})}{-U''_1(\hat{c})} \left( \frac{U_0'\hat{c})}{U_1'(\hat{c})} - 1 + \frac{-U''_1(\hat{c}) c}{U_1'(\hat{c})} \right) \quad (71a)
\]
\[
\hat{U}'(c) = U_1'(\hat{c}) \quad (71b)
\]
as we vary \( \hat{c} \). By condition (19), \( c > 0 \). \( U_1' > 0 \), thus \( \hat{U}' > 0 \), i.e. \( \hat{U} \) is strictly increasing. Because \( F(c) \) is weakly and \( \frac{U_1'}{U_1''} \) is strictly increasing, the right hand of (71a) implies that \( c \) is strictly increasing in \( \hat{c} \). Moreover, by concavity of \( U_1 \), the right hand side of (71b) is strictly decreasing in \( \hat{c} \). Therefore, \( \hat{U}'(c) \) is strictly decreasing in \( c \). So \( \hat{U} \) is strictly concave.
By the Envelope Theorem, the derivative of the left hand side of (70) is \( c \), where \( \hat{U}'(c) = p \).
The derivative of the right hand side is
\[
U_0'(\hat{c}) \frac{1}{U''_1(\hat{c})} - \hat{c} - \frac{U_1'(\hat{c})}{U''_1(\hat{c})},
\]
where \( U_1'(\hat{c}) = p \). Because of the second equation above, \( \hat{c} = \hat{c} \). After algebra manipulations, the first equation implies that the derivative of the right hand side of (70) equals to the derivative of the right hand side. By adding an appropriate constant to \( \hat{U}(c) \), we can construct \( \hat{U} \) that satisfies the properties in the lemma.

\[\square\]
Lemma 22. There exists $\hat{U}_0(c)$ strictly increasing and weakly concave, such that, for all $p > 0$.

$$\max_{0 \leq c \leq y} \{ \hat{U}_0(c) - pc \} = U_0(\hat{c}) - p\hat{c}$$  \hspace{1cm} (72)

where $\hat{c} = \arg \max_{c \leq y} U_1(c) - pc$.

Proof. Let $\hat{U}_0$ be defined as follow. $\hat{U}_0(c) = \hat{U}(c)$ for $c \leq c^* = \frac{U_1'(y)}{-U_1''(y)} \left( \frac{U_0'(y)}{U_1'(y)} - 1 + \frac{-U_0''(y)}{U_1''(y)} \right)$. By Assumption 1, $c^* \leq y$. For $c \in [c^*, y]$, $\hat{U}_0(c) = \hat{U}(c^*) \frac{y-c}{y-c^*} + U_0(y) \frac{c-c^*}{y-c}$. For $p > U_1'(y)$, $\hat{c} < y$, by Lemma 21 and the definition of $\hat{U}_0$, equality (72) holds. For $p \leq U_1'(y)$, by definition of $\hat{U}_0$:

$$U_0(y) - U_1'(y)y = \hat{U}(c^*) - U_1'(y)c^*.$$ 

Therefore, for $c \in [c^*, y]$:

$$\hat{U}_0'(c) = \frac{U_0(y) - \hat{U}(c^*)}{y-c} = U_1'(y) = \hat{U}'(y).$$

Because of the concavity of $\hat{U}$: $\hat{U}_0'(c) \geq \hat{U}'(y) \geq p$ for all $c$. Thus,

$$\max_{0 \leq c \leq y} \{ \hat{U}_0(c) - pc \} = \hat{U}_0(y) - py = U_0(y) - py,$$

which is the right hand side of (72). \qed

Proof of Theorem 11. We define $\hat{U}(c, a) = \hat{U}(c)$ for $a > 0$ and $\hat{U}(c, 0, y) = \hat{U}_0(c, y)$, where $\hat{U}_0(c, y)$ is defined in Lemma 22. Equations (70) and (72) imply that, $W$ satisfies

$$\rho W = \max_{c} \hat{U}(c, a) + W'(a)(ra + y - c) + \frac{1}{2} \nu^2 W''(a),$$

with the additional constraint $c \leq y$ when $a = 0$. This is the HJB equation for the value function, $\hat{V}$, of $\hat{U}$ consumer. Therefore $W = \hat{V}$. \qed
L Proofs for Equilibria under General Equilibrium

To prove Theorem 13, we first rewrite system (23) as an implicit ODE. In particular, \( V' (a) > 0 \) is determined implicitly as the solution \( p \) of the equation:

\[
(\rho + \lambda) V (a) - \lambda W (a) = H_f (p, a),
\]

(73)

where

\[ H_f (p, a) \equiv \max c U_1 (c) + p (f (a) - c). \]

(74)

The following lemma is the counterpart of Lemma 3 for exogenous return.

**Lemma 23.** When \( U_1 (f (a)) < (\rho + \lambda) V (a) - \lambda W (a) < \lim_{c \to \infty} U_1 (c) \), equation (73) has exactly two solutions \( p_1, p_2 \) with \( 0 < p_1 < U'_1 (f (a)) < p_2 \).

**Proof.** The same as the proof of Lemma 3.

**Proof of Theorem 13.** Using Lemma 23, Theorem 13 is proved in exactly the same way as Theorem 4 including the important Lemmas 12, 16, 15. For example, below we formulate and prove the counterpart of Lemma 12, Lemma 24, for this case with decreasing return. Starting from \( a^{ss} \), with the initial condition \( (V (a^{ss}), W (a^{ss})) = (V (a^{ss}), W (a^{ss})) \), we extend the system (23) to the left using the lower root for \( V' (a) \). Assume that at \( \hat{a} \), the system becomes singular. By a lemma similar to Lemma 15, we also have \( V (a) \geq V (a^{ss}) \) for all \( a \geq \hat{a} \). Therefore if \( \hat{a} > 0 \), \( V (\hat{a}) = V^{ss} \) and we must have

\[
\bar{V}' (a) \leq U'_1 (f (\hat{a})),
\]

or equivalently,

\[
\left( \frac{1}{\rho + \lambda} + \frac{\lambda}{(\rho + \lambda) \rho} \beta (f (\hat{a})) \right) f' (\hat{a}) \leq 1.
\]

(75)

This inequality immediately implies that \( \hat{a} < f^{-1} (\bar{c}) < a^{ss} \). Because for any \( a \in [f^{-1} (\bar{c}), a^{ss}) \),

\[
\left( \frac{1}{\rho + \lambda} + \frac{\lambda}{(\rho + \lambda) \rho} \beta (f (a)) \right) f' (a) = \frac{1}{\rho} f' (a) > \frac{1}{\rho} f' (a^{ss}) = 1.
\]

If there exists no \( \hat{a} > 0 \) such that inequality (75) is satisfied, then \( \hat{a} = 0 \), i.e. starting from any asset level below the steady-state, the decision maker saves up to reach the steady state in finite time.

**Lemma 24.** Starting at \( a^{ss} \) and \( (V (a^{ss}), W (a^{ss})) = (V (f (a^{ss})), W (f (a^{ss}))) \), extending the solution \( (V, W) \) downward following system (8), whenever \( f (a) - \bar{c} (a) \) stays strictly positive we
Given

1) \((\rho + \lambda - f'(a)) V'(a) - \lambda W'(a) < 0\)
2) \(V''(a) < 0\)
3) \(V'(a) > W'(a)\).

Proof. First, differentiating (23a), we obtain

\[
(\rho + \lambda - f'(a)) V'(a) - \lambda W'(a) = V''(a) (f(a) - \hat{c}(a))
\]  

(76)

Given \(f(a) - \hat{c}(a) > 0\), \((\rho + \lambda - f'(a)) V'(a) - \lambda W'(a) < 0\) if and only if \(V''(a) < 0\).

By the standard analysis for the neoclassical growth model, in the left neighborhood of \(a^{ss}\), we have \((\rho + \lambda - f'(a)) V'(a) - \lambda W'(a) < 0\) because \(V'(a) = W'(a)\) and \(f'(a) > f'(a^{ss}) = \rho\). Second, given that \(0 \leq \hat{c}(a) < f(a)\), \(V'(a) = U'_1(\hat{c}(a)) > U'_1(f(a)) > 0\), as long as \(V'(a)\) is defined.

Using these observations, we prove the first result by using Lemma 6. Condition 1) of Lemma 6 (in the left neighborhood of \(a^{ss}\)) is shown above. We now verify condition 2. If there exists \(\hat{a} < a_u\) such that \((\rho + \lambda - f'(\hat{a})) V'(\hat{a}) - \lambda W'(\hat{a}) = 0\). From equation (76) at \(\hat{a}\), \((\rho + \lambda - f'(\hat{a})) V'(\hat{a}) - \lambda W'(\hat{a}) = V''(\hat{a}) (f(\hat{a}) - \hat{c}(\hat{a})).\) This implies \(V''(\hat{a}) = 0\). Notice that \(U'_1(\hat{c}(a)) = V'(a)\) so \(U''_1(\hat{c}(a)) \hat{c}'(a) = V''(a)\) or

\[
\hat{c}'(a) = \frac{V''(a)}{U''_1(\hat{c}(a))}.
\]  

(77)

Besides, differentiating the second equation, (8), in the system (23) and using (77),

\[
\rho W'(a) = \frac{U'_0(\hat{c}(a))}{U''_1(\hat{c}(a))} V''(a)
\]

\[
+ W''(a) (f(a) - \hat{c}(a)) + W'(a) \left(f'(a) - \frac{1}{U''_1(\hat{c}(a))} V''(a) \right).
\]  

(78)

Since \(V''(\hat{a}) = 0\),

\[
\rho W'(\hat{a}) = W''(\hat{a}) (f(\hat{a}) - \hat{c}(\hat{a})) + W'(\hat{a}) f'(\hat{a}).
\]

Equivalently,

\[
W''(\hat{a}) = \frac{(\rho - f'(\hat{a})) W'(\hat{a})}{f(\hat{a}) - \hat{c}(\hat{a})} = \frac{(\rho - f'(\hat{a})) \rho + \lambda - f'(\hat{a})}{\lambda} \frac{V'(\hat{a})}{f(\hat{a}) - \hat{c}(\hat{a})} < 0.
\]
So

$$\lambda W''(\tilde{a}) < (\rho + \lambda - f'(\tilde{a})) V''(\tilde{a})$$.

Given that the two conditions of Lemma 6 are satisfied, this lemma implies the first property. The second and third properties are proved in exactly the same way as in Lemma 12.