Petty Envy When Assigning Objects

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September 15, 2017

Abstract

Envy of another person's assignment is "justified" if you "deserve" the object *and* it is possible to assign you to the object. Currently, the literature only considers whether or not the agent deserves the object and ignores whether or not assigning her to it is possible. This paper defines a fair set of assignments in terms of what is possible. We prove that there exists a unique fair set of assignments. The fair set of assignments has the same properties as the set of stable matches: the Lattice Theorem, Decomposition Lemma, and Rural Hospital Theorem all hold. Moreover, there is a unique, Pareto efficient, fair assignment: the assignment made by Kesten's Efficiency Adjusted Deferred Acceptance mechanism when all students consent.

Consider the problem of assigning children to schools. A key consideration is what constitutes a fair assignment. Determining what is "fair" is more than just a theoretical question; fairness has been the primary objective for most school boards, and it typically comes at the cost of efficiency. For example, New York City implements what is deemed to be a fair assignment, and as a result, it would be possible to improve approximately 6% of the student assignments without harming any of the other students (Abdulkadiroglu et al, 2009).

Typically, the literature calls an assignment unfair if there is a student who has *justified envy*. Student i envies student j if she strictly prefers j's assignment to her own. This envy is defined

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to be justified if i has higher priority than j at j's assignment.[1] Aristotle (in Rhetoric) defined envy as "the pain caused by the good fortune of others". Perhaps this envy is justified if that good fortune is rightfully yours. But is it justified to envy what you can never have? Most people would say no; this type of envy is petty.

Why is this relevant to school assignment? Suppose student i has justified envy of student j at school a. What if it is impossible to assign i to a but it is possible to assign j to a? Ignoring for the moment why it might be possible for one student and not for another, is it fair to not assign j to a just to keep i from being jealous? In this instance, we interpret i's objection as petty envy.

Consider the following classic example from Roth (1982) and applied to the school assignment problem by Abdulkadiroglu and Sonmez (2003).

Example 1. There are three students i, j, k, and three schools a, b, c, each of which has a capacity of one. The preferences and priorities are as follows:

R_i	R_j	R_k	$\succ_a$	$\succ_b$	$\succ_c$
b	a	a	i	j	j
a	b	b	k	i	i
c	c	c	j	k	k

We start with the assumption that a student with highest priority at an object should either be assigned to that object or an object she prefers to it. This implies that i must be assigned to b or a. Similarly, j must be assigned to a or b. Therefore, the only two assignments that are possible are the following:

$$\mu = \begin{pmatrix} i & j & k \\ a & b & c \end{pmatrix} \qquad \nu = \begin{pmatrix} i & j & k \\ b & a & c \end{pmatrix}$$

In particular, it is impossible to assign k to either a or b. Consider the assignment ν. k prefers a to her assignment. k has higher priority at a than j, the student assigned to a. Is k's envy of j "justified"? Our position is no. Since it is impossible to assign k to a, we view this envy as petty.

The key point is that we cannot determine if an assignment is fair in isolation; in order to know what assignments are fair, we must also know what alternative assignments are possible. Therefore, our

[1]This was introduced by Balinski and Sonmez (1999) as the definition of a fair assignment. Abdulkadiroglu and Sonmez (2003) introduced the term *justified envy*.

first objective is to determine which assignments are possible and which assignments are impossible. Central to our approach will be the concept of one assignment blocking another. We say that a student i blocks assignment μ with assignment ν if she prefers her assignment under ν and she has higher priority at that object than one of the students assigned to it under μ. We define an assignment to be impossible if it is blocked by a possible assignment. We define an assignment to be possible if every assignment that blocks it is impossible.

We use a recursive approach to show that this definition is well-defined. Assignments with no justified envy are not blocked by any assignment; therefore, they are possible. An assignment that is blocked by an assignment with no justified envy is therefore impossible. Determining that some assignments are impossible allows us to conclude that additional assignments are possible; if an assignment is only blocked by impossible assignments, it is possible. Determining that more assignments are possible allows us to determine that more assignments are impossible, and so on. Our main technical result is to demonstrate that this recursive approach classifies every assignment as either possible or impossible.

Using this definition of which assignments are possible and impossible, we introduce a new definition of fairness for school assignment. We say school a is possible (impossible) for student i if there exists a possible (impossible) assignment where she is assigned to a. Given an assignment μ, we define student i to have *legitimate envy* if she prefers a school a to her assignment, she has higher priority at school a then one of the students assigned to a, and it is possible to assign her to a. We define student i to have *petty envy* if she prefers school a to her assignment, she has higher priority at school a then one of the students assigned to a, but it is impossible to assign her to a. We define an assignment to be *unfair* if a student has legitimate envy and an assignment to be *fair* if any envy is petty.

Under this definition of fairness, we show that the set of possible assignments is the unique fair set of assignments. Moreover, we demonstrate that the classical properties that hold for the set of stable marriages in the marriage problem hold for the set of fair assignments. The Lattice Theorem, Decomposition Lemma, and the Rural Hospital Theorem all hold for the set of fair assignments. The Lattice Theorem implies that there is a well-defined, student-optimal fair assignment. We demonstrate a surprising result. The student-optimal fair assignment is Pareto efficient, and in particular, it is the assignment made by Kesten's efficiency adjusted deferred acceptance algorithm

(Kesten, 2010).[2]

This is the main conclusion of our paper. Fairness and efficiency are not incompatible. So long as we do not honor petty envy, there is a unique fair and Pareto efficient assignment.[3]

1 Relationship to the Literature

There are a number of school assignment papers that have considered alternative interpretations of fairness. Kesten (2004) defines an assignment to be reasonably fair if whenever student i has justified envy of a school a, then there is no fair assignment that assigns i to a. Kesten introduces several algorithms, including EADA, that Pareto improve the DA assignment and are reasonably fair. Alcalde and Romero (2015) consider a fairness notion closely related to reasonable fairness. They allow a student i to block using school a if there is an assignment with no justified envy in which i is assigned to a.[4] They call unblocked assignments α-equitable and show that an assignment is α-equitable if and only if it weakly Pareto dominates an assignment with no justified envy. While α-equity is similar in spirit to our definition, there are important differences. Most importantly, the set of α-equitable assignments is unfair (using our definition) in the sense that one α-equitable assignment may block another. This can be seen in our Example 4. The fairness concept that is closest to ours is essentially stable introduced by Kloosterman and Troyan (2016). We refer the reader to their paper for a formal definition of essentially stable, but intuitively, an assignment is essentially stable if whenever a student i has justified envy of school a, placing i at

[2] More precisely, the assignment made by the efficiency adjusted deferred acceptance algorithm when all student's consent.

[3] We do not discuss in this paper whether or not it is possible to implement the Kesten assignment. There is reason to be pessimistic on this front. Kesten 2010 has a full discussion on the topic, but he proves that it is impossible for a strategyproof mechanism to implement his assignment. Dur and Morrill (2016) prove that no manipulable assignment implements the Kesten assignment even in a Nash equilibrium.

[4] This is not the way α-equity is defined in Alcalde and Romero (2015), but it is equivalent to their definition. Specifically, they define (i, μ') to be an ϵ-objection to μ if $\mu'_i \; P_i \; \mu_i$ and $i \succ_{\mu'_i} j$ where $\mu_j = \mu'_i$. An ϵ-objection (i, μ') is admissible if no student has an ϵ-objection to μ'. An assignment μ is α-equitable if there is no admissible ϵ-objection to μ. An ϵ-objection is equivalent to an assignment being blocked by another assignment. Since any assignment with justified envy can be blocked, only an assignment with no justified envy has no counter objections. Therefore, an assignment is α-equitable if it is not blocked by any assignment that eliminates justified envy. Reasonable stability, introduced by Cantala and Papai (2014), is closely related to α-equity.

a and removing a student leads to a succession of appeals that ultimately leads to the removal of student i from school a. Kloosterman and Troyan (2016) and the current paper were developed independently. Essentially stable and petty envy are similar in spirit, but Kloosterman and Troyan (2016) demonstrate that the two concepts are not equivalent.

Our notion of petty envy is in the spirit of bargaining sets introduced by Zhou (1994). For a transferable utility, cooperative game, he introduces bargaining sets as a generalization of the core. Specifically, a coalition can block with an alternative only if that alternative is not subsequently blocked. He demonstrates that the bargaining set is non-empty for every transferable-utility game. Most closely related to our paper is the literature on von Neumann-Morgenstern Stable sets (hereafter, vNM stable sets). We use the same concept of internal and external stability in our definition; however, our definition of blocking is different than under vNM stability. Interestingly, several papers have consider vNM stability in regards to the marriage problem[5] and have found similar results. Ehlers (2007) characterizes vNM stability for the marriage problem by showing that any vNM stable set V is a maximal set satisfying the following properties: (a) the core is a subset of V, (b) V is a distributive lattice, and (c) the set of unmatched students is the same for all matchings belonging to V. Ehlers remarks that his characterization does not extend to the many-to-one matching problem and that there need not be any relationship between vNM stable sets of a many-to-one problem and vNM stable sets of its corresponding one-to-one matching problem.[6] Motivated by Ehlers (2007), Wako (2010) introduces an algorithm that proves that there exists a unique vNM stable set for any marriage game.[7] Bando (2014) proves that in the marriage problem, EADA produces the male-optimal match in the unique vNM stable set.

Our approach can be viewed as dividing priorities into those that must be honored and those that can be disregarded. A paper that considers a similar notion is Dur, Gitmez, and Yilmaz (2015). Intuitively, they define an assignment to be partially fair if the only priorities that are violated

[5] A marriage problem is a one-to-one matching problem. In our framework, it is equivalent to each school having a capacity of one.

[6] From the working paper Ehlers (2005), it is clear that he means that there is no direct relationship between the vNM stable set of a many-to-one matching problem and the vNM stable set of the "clone" economy made by creating q_a clones of each school a and modifying preferences in a specific way. The cloning procedure is useful for many problems, but in this instance it creates artificial structure that alters the set of vNM stable assignments.

[7] Wako actually showed existence and uniqueness in an earlier note, Wako (2008). Our results were all discovered independently of Wako's work, but the approach he takes in Wako (2008) is similar to the mathematical approach we have taken in this paper. His work does not directly apply to our paper as it is restricted to one-to-one matches.

are "acceptable violations". The key difference between petty envy and partial fairness is that the priority violations deemed petty are endogenous to our model whereas the set of "acceptable violations" for partial fairness is exogenously given. Dur et al. (2015) also provide a striking result regarding EADA. They show that the unique mechanism that is constrained efficient, no-consent proof,[8] and Pareto improves the DA assignment is EADA. We show that EADA is the unique efficient and fair assignment. Their conditions are properties of a mechanism while our conditions are properties of an assignment, so our results do not relate to each other directly; nonetheless, there is clearly a strong and complementary relationship between the two results.

2 Model

There is a set of students, $A = \{i, j, k, \dots\}$, to be assigned to a set of schools, $O = \{a, b, c, \dots\}$. Each student i has strict preferences P_i over the schools. We allow students to be unassigned. $i\ P_i\ a$ indicates that student i prefers being unassigned to being assigned to school a. Each school a has priorities $\succ_a$ over the students. R and $\succeq$ represent the weak preferences and priorities corresponding to P and $\succ$, respectively. Each school has a maximum number of students that it can be assigned. We let q_a denote the maximum capacity of school a. For simplicity, we assume that every student is acceptable to every school although this assumption is not necessary for any of our results to hold.

An assignment μ is a function from students to schools such that no school a is assigned more than q_a students. $\mu_i = a$ indicates that student i is assigned to school a. In a slight abuse of notation, for each school a we set $\mu_a = \{i : \mu_i = a\}$. $\mu_i = i$ indicates that student i is left unassigned.

An assignment μ is **individually rational** if for every student i, $\mu_i\ R_i\ i$. An assignment μ is **wasteful** if there is a student i and a school a such that $aP_i\mu_i$ and $|\mu_a| < q_a$. In this paper we consider only individually rational and nonwasteful assignments as we assume that these are the only assignments that would be considered by a policy maker.[9]

[8]See Dur et al. (2015) for a formal definition of no-consent proof, but intuitively a mechanism is no-consent proof if no student is harmed by allowing her priority at a given object to be violated.

[9]This assumption is not without loss of generality. Ehlers and Morrill (2017) shows that if wasteful assignments are possible than the Strong Rural Hospital Theorem no longer holds. However, the other results and all conclusions of the paper continue to hold.

Given an assignment μ, student i has **justified envy** if there exists a school a such that (i) $aP_i\mu_i$ and (ii) there exists a $j \in \mu_a$ such that $i \succ_a j$.

3 Petty Envy

3.1 Which assignments are possible?

Our first goal is to determine which school assignments are impossible and which assignments are possible. Central to our approach will be the notion of a student blocking an assignment with an alternative assignment.

Definition 1. Student i **blocks** assignment μ with assignment ν if for some school a: (1) $aP_i\mu_i$; (2) $i \succ_a j$ where $\mu_j = a$; and (3) $\nu_i = a$. We also say μ is blocked by ν if there is a student i who blocks μ with ν.

We define an assignment to be impossible if it is blocked by a possible assignment. We define an assignment to be possible if it is either unblocked or only blocked by assignments that are impossible.

Definition 2. An assignment μ is **impossible** if it is blocked by a possible assignment. An assignment μ is **possible** if any assignment that blocks it is impossible.

It is not obvious that it is necessarily possible to partition the set of assignments into possible and impossible ones (as suggested by this definition). Our main result is to show that not only is such a delineation possible, but it is unique. We will use the following function repeatedly, so we define it explicitly. Given any set of assignments S, $\pi(S)$ is the set of assignments that are not blocked by any assignment in S.

$$\pi(S) = \{\mu | \nexists \nu \in S \text{ such that } \nu \text{ blocks } \mu\} \tag{1}$$

The following two facts are useful:

Monotonicity: If $A \subseteq B$, then $\pi(B) \subseteq \pi(A)$.

Justified Envy: If J are the assignments with no justified envy, then $J \subseteq \pi(A)$ for any set of assignments A.

Assignments with no justified envy is not blocked by any assignment, impossible or otherwise; therefore, they are possible. We refer to this set as "too small" as we anticipate finding additional possible assignments.

$$S^1 := \text{assignments with no justified envy.}$$

We are defining an iterative process. We next define:

$$B^1 = \pi(S^1)$$

The assignments that eliminate justified envy (S^1) are possible. An assignment blocked by a possible assignment is impossible. Therefore, each assignment $\mu \notin B^1$ is impossible. However, we anticipate that B^1 is likely "too big" as we expect to find that some of the assignments in B^1 are impossible.

In general, for any $k > 1$ we define the small and big sets as follows:

$$S^k = \pi(B^{k-1}) \tag{2}$$

$$B^k = \pi(S^k) \tag{3}$$

Consider the set $S^2 = \pi(B^1)$ and let $\mu \in S^2$. If $\mu \in S^1$, then μ is possible. If $\mu \in S^2 \setminus S^1$, then every assignment that blocks μ is impossible (specifically, if ν blocks μ, then $\nu \notin B^1$ by definition). Therefore, each assignment in S^2 is possible. Similarly, any assignment $\mu \notin B^2$ is impossible, and so on. A key point is that the number of assignments we have determined to be possible is weakly increasing.

Lemma 1. *For every $k > 1$, $S^{k-1} \subseteq S^k$.*

Proof. Our proof is by induction. $S^1 \subseteq S^2$ by the No Justified Envy Fact. Now consider any $k > 1$. By the inductive hypothesis, $S^{k-2} \subseteq S^{k-1}$. Therefore, by the Monotonicity Fact, $\pi(S^{k-1}) =$

$B^{k-1} \subseteq \pi(S^{k-2}) = B^{k-2}$. Since $B^{k-1} \subseteq B^{k-2}$, again by the Monotonicity Fact, $\pi(B^{k-2}) = S^{k-1} \subseteq \pi(B^{k-1}) = S^k$. $\square$

As there are only a finite number of assignments, eventually the process must terminate. Let n be the first integer such that $S^n = S^{n+1}$. Our next theorem establishes that at the conclusion of the iterative process, every assignment has been determined to be either possible or impossible.

Theorem 1. *$S^n = B^n$.*

We prove Theorem 1 in the Appendix.

We consider an alternative definition of possible assignments and show that the two definitions coincide. Intuitively, assignments that are unblocked should be possible and assignments that are blocked should be impossible.

Definition 3. P is a **possible set of assignments** if (i) for any $\mu, \nu \in P$, μ does not block ν; and (ii) if $\mu \notin P$, then there exists a $\nu \in P$ such that ν blocks μ.

The definition of a possible set of assignments is in the spirit as vNM stability, but the two concepts are logically different. vNM stability is defined using the same two conditions: a set of assignments A is vNM stable if (i) for any $\mu, \nu \in A$, μ does not block ν; and (ii) if $\mu \notin A$, then there exists a $\nu \in A$ such that μ blocks ν. However, the our definition of blocking is different than that used for vNM stability. vNM stability is a generalization of the core, and it requires that no agent in a coalition is worse off and some agent in the coalition is strictly better off. In our context, we allow student i to block assignment μ with assignment ν if she has higher priority at $a = \nu_i$ than some student $j \in \mu_a$. vNM stability requires that school a prefers ν_a to μ_a (in fact vNM stability is even stronger as it also would require that for every $k \in \nu_a \setminus \mu_a$, $aP_k\mu_k$). Note that in our model, as is typical in a real world assignment problem, the schools do not have preferences over assignments but instead have a priority ranking of students. Therefore, it is not even well-defined to say whether or not school a prefers one assignment to another. Even if each school has responsive preferences[10] over assignments, the following definition illustrates that the two notions of blocking are not equivalent.

[10]School a's preferences P_a are **responsive** to priorities $\succ_a$ if for every set of students $A' \subseteq A$ such that $|A'| < q_a$ and any students $i, j \in A \setminus A'$, then $A' \cup \{i\} \, P_a \, A' \cup \{j\}$ if and only if $i \succ_a j$.

Example 2. Suppose there are four students $\{i, j, k, l\}$, two schools $\{a, b\}$, and that each school has a capacity of two. Further, suppose that all students prefer a to b. Let school a have priorities $l \succ_a i \succ_a j \succ_a k$. Moreover, suppose a's preferences over assignments with two students are as follows: $\{l, i\} \, P_a \, \{l, j\} \, P_a \, \{l, k\} \, P_a \, \{i, j\} \, P_a \, \{i, k\} \, P_a \, \{j, k\}$. It is straightforward to verify that these preferences are responsive to a's priorities. Consider two assignments μ and ν such that $\mu_a = \{l, j\}$ and $\nu_a = \{i, k\}$ (under each, the remaining students are assigned to b). Note that aP_ib, $i \succ_a j$, and $\nu_i = a$. Therefore, i blocks μ with ν under our definition of blocking. However, $\mu_a P_a \nu_a$, so the coalition of $\{i, k, a\}$ does not block μ with ν.

Note that if each school has a capacity of one, then the definition of a possible set of assignments and vNM stability are equivalent. Indeed, in that case, our results mirror the results found in Ehlers (2007), Wako (2008), Wako (2010), and Bando (2014). However, to the best of our knowledge, no results for vNM stability are known for the general case considered here where schools may have capacity for more than one student.

As Theorem 2 demonstrates, we can immediately conclude that S^n is the unique possible set of assignments. Therefore, we will now refer to S^n as the set of possible assignments.

Theorem 2. *S^n is the unique possible set of assignments.*

Proof. (of Theorem 2) By Theorem 1, $S^n = B^n = \pi(S^n)$. Therefore, it is straightforward to verify that $\mu \notin S^n$ if and only if μ is blocked by a $\nu \in S^n$. Consequently, S^n is a possible set of assignments.

To show uniqueness, suppose A is a possible set of assignments. We first show that $A = \pi(A)$. No assignment $\mu \in A$ blocks an assignment $\nu \in A$. Therefore, $A \subseteq \pi(A)$. Consider $\mu \in \pi(A)$. If $\mu \notin A$, then by condition (ii) μ is blocked by a $\nu \in A$. Therefore, $\mu \notin \pi(A)$. Since $\mu \notin A$ implies $\mu \notin \pi(A)$, $\pi(A) \subset A$, and indeed, $A = \pi(A)$. An assignment with no justified envy must be possible, therefore, $S^1 \subseteq A$. By the Monotonicity Fact, $\pi(A) \subseteq \pi(S^1)$. Since $\pi(A) = A$ and $\pi(S^1) = B^1$, $A \subseteq B^1$. Again, by the monotonicity fact, $\pi(B^1) \subseteq \pi(A)$ implying $S^2 \subseteq A$. Iterating this argument, we get that for any k, $S^k \subseteq A \subseteq B^k$. Since $S^n = B^n$, it must by that $S^n = A$. $\square$

3.2 Petty Envy

Now that we have uniquely determined which assignments are possible and which assignments are impossible, we are ready to introduce our notion of fairness.

Definition 4. Given an assignment μ, student i has **legitimate envy** of school a if i has justified envy of a and it is possible to assign i to a. Student i has **petty envy** of school a if i has justified envy of a and it is impossible to assign i to a.

We define an assignment μ to be **fair** if no student has legitimate envy. Note that an assignment is fair if and only if it is possible.

3.3 Lattice Structure

The set of possible assignments is a superset of the set assignments that eliminate justified envy. But we show that the two sets have the same mathematical structure. Specifically, the Lattice Theorem, Decomposition Lemma, and Rural Hospital Theorem all hold. To emphasize the connection with classical matching theory, our Lemmas and proofs mirror the presentation in Roth and Sotomayor (1990).

Our process is a modification of the "cloning" procedure used by Gale and Shapley (1962) and covered extensively in Roth and Sotomayor (1990). If school a has a capacity of q, then we create q **seats** at a, $a^1, \ldots, a^q$. Each seat at a has the same priorities over students as the school a. Unlike the cloning procedure, we do not define any student preferences over seats. A student only has preferences over schools. A school assignment assigns each student to a school. A **seat assignment** assigns each student to a seat at a school. Given a school assignment μ, a seat assignment $\overline{\mu}$ is a **μ-seat assignment** if for each student i, $\overline{\mu}_i$ is a seat at μ_i. For expositional ease, the seat assignment $\overline{\mu}$ is understood to be a μ-seat assignment.

Definition 5. Given two school assignments μ and ν, $\overline{\mu}$ and $\overline{\nu}$ are **consistent seat assignments** if for any student i such that $\mu_i = \nu_i$, $\overline{\mu}_i = \overline{\nu}_i$.

In words, if i receives the same assignment under μ and ν, then i is assigned to the same seat under $\overline{\mu}$ and $\overline{\nu}$. We will work exclusively with consistent seat assignments. Given two seat assignments,

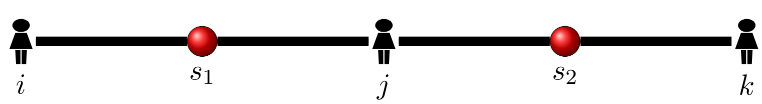
$\overline{\mu}$ and $\overline{\nu}$, we induce a graph $G^{\overline{\mu},\overline{\nu}}$ as follows (when $\overline{\mu}$ and $\overline{\nu}$ are clear from context, we will refer to the graph as G). Each student and each seat is a vertex. There is an edge between student i and seat s if i is assigned to s under either $\overline{\mu}$ or $\overline{\nu}$.[11] Each vertex has degree less than or equal to two; therefore, each connected component of the graph is either a path or a cycle.

Now, suppose that μ and ν are individually rational, nonwasteful, and do not block each other. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments and let $G = G^{\overline{\mu},\overline{\nu}}$. We will define **pointing** analogously to the discussion in Roth and Sotomayor (1990). We ask each student i to point at the seat of her favorite school (between μ_i and ν_i). Similarly, if a seat is assigned to different students under μ and ν, then we ask it to point at the student with highest priority.

Lemma 2 (Pointing Lemma)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other, and let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. Then (1) no two students point to the same seat, and (2) no student and seat point to each other (unless the student receives the same assignment under μ and ν).*

Proof. Suppose for contradiction that both i and j point at seat s where s is a seat at school a. Without loss of generality, $i \succ_a j$ and $\overline{\mu}_i = s$ (and consequently that $\overline{\nu}_j = s$). Therefore, $\mu_i = a$ and $\nu_j = a$. Note that $\mu_i P_i \nu_i$, $i \succ_{\mu_i} j$, and $\nu_j = \mu_i$. Therefore, i blocks ν with μ, a contradiction. Similarly, suppose i points at seat s and seat s points to i but $\mu_i \neq \nu_i$. Let s be a seat at school a (i.e. $\mu_i = a$). i prefers μ_i to ν_i. a must be assigned to capacity by ν or else ν would be wasteful. Let j be the student assigned to seat s. Since s points at i, $i \succ_a j$. But since $aP_i\nu_i$ and $i \succ_a j$ where $\nu_j = a$, i blocks ν with μ, a contradiction. □

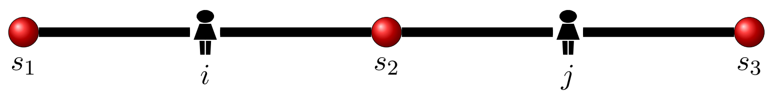
Potentially, a path between students and seats could occur three different ways: each endpoint of the path is a student, each endpoint is a school, or else one endpoint is a student and one endpoint is a school. Consider the path below where each endpoint is a student (more generally, a path $\{i_1, s_1, i_2, s_2, \ldots, s_{n-1}, i_n\}$).



[11]If $\mu_i = \nu_i = i$, then we draw an edge from i to itself. Similarly, if j is assigned to the same seat s under both $\overline{\mu}$ and $\overline{\nu}$ then we draw two edges between j and s. We refer to both instances as trivial cycles.

Student i must point at seat s_1 (otherwise, i prefers being unassigned to s_1 and the assignment of i to s_1 is not individually rational). Similarly, k must point at s^2. Therefore, whichever seat j points to has two students pointing at it, contradicting the Pointing Lemma. More generally, n students, $\{i_1, \ldots, i_n\}$, point at $n-1$ seats, $\{s_1, \ldots, s_{n-1}\}$. By the pigeon hole principle, two students must be pointing at the same seat which is a contradiction.

Suppose instead that the endpoints are seats such as the path below (more generally, a path $\{s_1, i_1, s_2, i_2, \ldots, i_{n-1}, s_n\}$).



Note that when i is assigned to s_2, s_1 is unassigned. Therefore, i cannot point at s_1 or else it would be wasteful to leave s_1 unassigned. Similarly, j cannot point at s_3. Therefore, both i and j point at s_2, a contradiction. More generally, the $n-1$ students, $\{i_1, \ldots, i_{n-1}\}$, point to $n-2$ schools $\{s_2, \ldots, s_{n-1}\}$ implying that two students must point at the same school, a contradiction.

Finally, consider the path below where one endpoint is a student and one endpoint is a school (more generally (more generally, a path $\{i_1, s_1, i_2, \ldots, i_n, s_n\}$).

Repeating the logic of the previous two cases, i must point at s_1 or else her assignment to s_1 would not be individually rational. j cannot point to s_2 or else it would be wasteful to leave s_2 unassigned. But this implies that i and j both point at s_1, a contradiction. More generally, since n students, $\{i_1, \ldots, i_n\}$ point at $n-1$ schools, $\{s_1, \ldots, s_{n-1}\}$, two students must point at the same seat which is a contradiction.

Therefore, we conclude that there are no paths in our graph. Each partition of students and seats must be a cycle. We summarize this in the following lemma.

Lemma 3. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. Moreover, let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. Then each student i_1 is part of a unique cycle $\{i_1, s_1, \ldots, i_n, s_n\}$ where $\overline{\mu}_{i_k} = s_k = \overline{\nu}_{i_{k+1}}$.*[12]

[12] It is understood that $i_{n+1} = i_1$. As a reminder, if $\mu_i = \nu_i$, then we have constructed i to be part of a trivial cycle.

We will refer to $\{i_1, s_1, \ldots, i_n, s_n\}$ as i_1's $(\overline{\mu}, \overline{\nu})$**-cycle**. Note that different seat assignments produce potentially different cycles. We give an example to aid with intuition.

Example 3. Let there be three schools, a, b, and c, and eight students $\{i_1, i_1, \ldots, i_8\}$. Schools a and b have a capacity of 2 and school c has a capacity of 3. Define assignments μ and ν by

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & a & a & b & b & b & c & c \end{pmatrix} \quad \nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & b & c & a & b & c & a & b \end{pmatrix}$$

Note that i_1 and i_5 are the only students that receive the same assignment under μ and ν. Therefore, our only restriction on a consistent seat assignment is that i_1 and i_5 must each be assigned to the same seat. In the table below, $\overline{\mu}$ and $\overline{\nu}$ are consistent seat assignments. $\overline{\mu}$ and $\overline{\nu}'$ are also consistent seat assignments. However, $\overline{\mu}$ and $\overline{\nu}''$ are not consistent seat assignments because i_5 is assigned to a different seat at b under $\overline{\mu}$ and $\overline{\nu}''$.

$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$	$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$	$\overline{\mu}$		$\overline{\nu}$	$\overline{\nu}'$	$\overline{\nu}''$
i_1	a^1	i_1	i_1	i_1	i_4	b^1	i_2	i_8	i_5	i_7	c^1	i_3	i_3	i_3
i_2	a^2	i_4	i_7	i_7	i_5	b^2	i_5	i_5	i_8	i_8	c^2	i_6	i_6	i_6
i_3	a^3	i_7	i_4	i_4	i_6	b^3	i_8	i_2	i_2					

In this example, the $(\overline{\mu}, \overline{\nu})$-cycles are (i_1, a^1), (i_5, b^2), (i_2, a^2, i_4, b^1), (i_3, a^3, i_7, c^1), and (i_6, b^3, i_8, c^2). The $(\overline{\mu}, \overline{\nu}')$-cycles are (i_1, a^1), (i_5, b^2), and $(i_2, a^2, i_7, c^1, i_3, a^3, i_4, b^1, i_8, c^2, i_6, b^3)$.

An immediate consequence of Lemma 3 is our version of the Rural Hospital Theorem.

Theorem 3 (Rural Hospital)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. A student i is assigned under μ if and only if i is assigned under ν. Moreover, every school is assigned the same number of students under μ and ν.*

Proof. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. By Lemma 3, every student who receives an assignment is part of a unique cycle. This implies that every student and every seat is either assigned under both μ and ν or under neither. $\square$

The Pointing Lemma tells us that a student and a seat cannot point at each other. This implies our version of the Decomposition Lemma. We illustrate this in Figure 1. Consider any student i_1 who

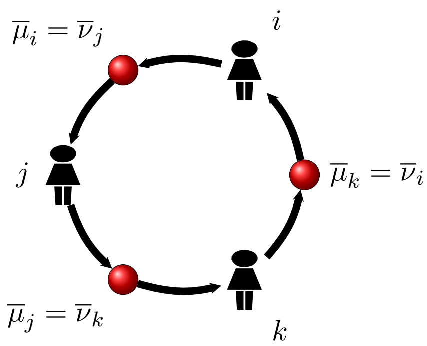


Figure 1: Every cycle must have a well-defined orientation. Either all students prefer μ and all schools "prefer" ν or else the opposite.

is part of a nontrivial cycle $\{i_1, s_1, \ldots, i_n, s_n\}$. If i_1 points at s_1, then s_1 must point at i_2. But if s_1 points at i_2, then i_2 must point to s_2, and so on. Similarly, if i_1 points at s_n, then s_n must point at i_n who must point at s_{n-1} and so on. The key point is that after learning the preferences of one student i, we can infer both the preferences of every student and the priorities of every seat in i's cycle. Similarly, learning for any one seat which student has higher priority implies the preferences and priorities of all other students in the cycle.

Lemma 4 (Decomposition Lemma)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. Moreover, let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments and let $\{i_1, s_1, \ldots, i_n, s_n\}$ be a nontrivial $(\overline{\mu}, \overline{\nu})$-cycle. Then either*

$$s_k \; P_{i_k} \; s_{k-1} \text{ and } i_{k+1} \succ_{s_k} i_k \text{ for every } 1 \leq k \leq n$$

or else

$$s_{k-1} \; P_{i_k} \; s_k \text{ and } i_{k-1} \succ_{s_{k-1}} i_k \text{ for every } 1 \leq k \leq n.$$

Note that we can now prove a strong version of the Rural Hospital Theorem (which is true for the college admissions problem as well). The Rural Hospital Theorem says that a school is assigned to the same number of students under any two individually rational, nonwasteful assignments that do not block each other. In fact, if a school ever is assigned to less than its capacity, then it receives exactly the same set of students under either assignment.

Theorem 4 (Strong Rural Hospital Theorem)**.** *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a, if $|\mu_a| < q_a$, then $\mu_a = \nu_a$.*

Proof. Suppose for contradiction that $|\mu_a| < q_a$ but that $\mu_a \neq \nu_a$. By the Rural Hospital Theorem, $|\mu_a| = |\nu_a|$, and in particular, $\mu_a \not\subseteq \nu_a$. Therefore, there exists a student i such that $\mu_i = a$ but $\nu_i \neq a$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. i is part of a unique $(\overline{\mu}, \overline{\nu})$-cycle, and in particular there exists a student j who is assigned to i's seat at a by $\overline{\nu}$. Specifically, $\overline{\mu}_i = \overline{\nu}_j$. i cannot prefer μ to ν or else ν is wasteful (a is not assigned to its capacity under ν). However, by the Decomposition Lemma, all students in the cycle prefer ν to μ. Since $\nu_j = a$, μ is wasteful as $aP_j\mu_j$ and a has available seats under μ. $\square$

We have only specified priorities at a school for individual students. Therefore, in general we cannot compare two different sets of students. However, we will show that when two assignments do not block each other, we can compare the set of students assigned to a school.

Definition 6. Consider a school a and two sets of students A and B such that $|A| = |B|$ but $A \neq B$. Then

$$A >_a B \Leftrightarrow \text{ for all } i \in A \setminus B \text{ and } j \in B \setminus A,\ i \succ_a j$$

We say that $A \geq_a B$ if either $A = B$ or else $A >_a B$.

Lemma 5. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a, if $\mu_a \neq \nu_a$, then either $\mu_a >_a \nu_a$ or else $\nu_a >_a \mu_a$.*

Proof. Label $\mu_a \setminus \nu_a = \{i_1, i_2, \ldots, i_n\}$ in decreasing order of priority (i.e. $i_k \succ_a i_{k+1}$). By the Rural Hospital Theorem, $|\nu_a \setminus \mu_a| = |\mu_a \setminus \nu_a|$. Label $\nu_a \setminus \mu_a = \{j_1, j_2, \ldots, j_n\}$ in decreasing order of priority. Without loss of generality, $i_n \succ_a j_n$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments such that i_n and j_n are assigned to a's first seat, a^1 ($\overline{\mu}_{i_n} = \overline{\nu}_{j_n} = a^1$). Then i_n and j_n are in the same $(\overline{\mu}, \overline{\nu})$-cycle. By the General Decomposition Lemma, since $i_n \succ_a j_n$, $\nu_{i_n} P_{i_n} \mu_{i_n} = a$.

Now define a new seat assignment, $\overline{\nu}'$ by switching j_n and j_1's seats under $\overline{\nu}$. In particular, $\overline{\nu}'_{j_1} = a^1 = \overline{\mu}_{i_n}$. It is straightforward to verify that $\overline{\mu}$ and $\overline{\nu}'$ are consistent seat assignments. Now i_n and j_1 are in the same $(\overline{\mu}, \overline{\nu}')$-cycle, but i_n's seat assignments have not changed. Therefore, i_n still points at the same seat, $\overline{\nu}'_{i_n}$. By the General Decomposition Lemma, every student in the cycle points to $\overline{\nu}'$, and every seat must point to $\overline{\mu}$. In particular, $i_n \succ_{a^1} j_1$. This proves the desired result since i_n is the student in $\mu_a \setminus \nu_a$ with the lowest priority at a and j_1 was the student in $\nu_a \setminus \mu_a$ with the highest priority at a. $\square$

We are now ready to prove the Lattice Theorem. Given two assignments μ and ν, we define the ordering $\geq$ by

$$\mu \geq \nu \text{ if for every student } i,\ \mu_i\ R_i\ \nu_i \text{ and every school } a, \nu_a \geq_a \mu_a \tag{4}$$

We define $\mu > \nu$ if the preference is strict for one of the students. Given two assignments μ and ν, define $\mu \vee \nu_i = \max_i \{\mu_i, \nu_i\}$. Define $\mu \wedge \nu_i = \min_i \{\mu_i, \nu_i\}$.

Lemma 6. *Let μ and ν be individually rational and nonwasteful assignments that do not block each other. For any school a,*

$$\mu \vee \nu_a := \{i | \mu \vee \nu_i = a\} = \min_{\geq_a} \{\mu_a, \nu_a\}$$

and

$$\mu \wedge \nu_a := \{i | \mu \wedge \nu_i = a\} = \max_{\geq_a} \{\mu_a, \nu_a\}.$$

Proof. If $\mu_a = \nu_a$, then the result is trivial. Now, suppose that $\mu_a \neq \nu_a$. By Lemma 5, either $\mu_a >_a \nu_a$ or else $\nu_a >_a \mu_a$. Without loss of generality, $\mu_a >_a \nu_a$. Consider any $i \in \nu_a \setminus \mu_a$. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments. i is part of a $(\overline{\mu}, \overline{\nu})$-cycle. Since $\mu_a >_a \nu_a$, $\overline{\nu}_i$ (i's seat at a under the seat assignment $\overline{\nu}$) does not point at i. Therefore, by the General Decomposition Lemma, i points to $\overline{\nu}_i$ and in particular $aP_i\mu_i$. Therefore, for any $i \in \nu_a \setminus \mu_a$, $\mu \vee \nu_i = a$ and $\mu \wedge \nu_i \neq a$. Now consider any $j \in \mu_a \setminus \nu_a$ (by the Rural Hospital Theorem, such a j exists). Since we know $j \succ_a i$, $\mu_i = a$, and $\nu_j \neq a$, it must be that $\nu_j P_j \mu_j = a$ or else j would block ν with μ. Therefore, for any $j \in \mu_a \setminus \nu_a$, $\mu \wedge \nu_i = a = \mu_i$ and $\mu \vee \nu_i \neq a$. So indeed, $\mu \vee \nu_a = \nu_a = \min_{\geq_a} \{\mu_a, \nu_a\}$ and $\mu \wedge \nu_a = \mu_a = \max_{\geq_a} \{\mu_a, \nu_a\}$. $\square$

We can immediately conclude from Lemma 6 that $\mu \vee \nu$ and $\mu \wedge \nu$ are well-defined assignments when μ and ν are individually rational and nonwasteful assignments that do not block each other. For any school a, by Lemma 5 either $\mu_a = \nu_a$, $\mu_a >_a \nu_a$, or else $\nu_a >_a \mu_a$. Therefore, $\min_{\geq_a}$ and $\max_{\geq_a}$ are well-defined sets, and since μ and ν are well-defined assignments, no more than q_a students are assigned to any school a.

We now prove the Lattice Theorem. From the construction in Section 3, we defined S^1 to be the assignments with no justified envy, and for any k, $B^k = \pi(S^k)$ (the assignments not blocked by any assignment in S^k), and $S^{k+1} = \pi(B^k)$. We fixed an n such that $S^n = S^{n+1}$. For our purposes, the following two points are critical. First, if $\mu, \nu \in S^n$, then μ and ν do not block each other.[13] Second, if $\mu \notin S^n$, then there exists a $\nu \in B^n$ such that ν blocks μ.[14]

Theorem 5 (Lattice Theorem)**.** *Let $\mu, \nu \in S^n$. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are in S^n.*

Proof. Let $\mu, \nu \in S^n$. As a reminder, by assumption we are only considering individually rational and nonwasteful assignments.[15] By construction, they do not block each other. Let $\lambda = \mu \vee \nu$ and let $\tau = \mu \wedge \nu$. Suppose for contradiction that $\lambda \notin S^n$. Then λ is blocked by some student i and some assignment $\lambda' \in B^n$. Specifically, for some school a $\lambda'_i = aP_i\lambda_i$ and there exists a $j \in \lambda_a$ such that $i \succ_a j$. By Lemma 6, $\lambda_a = \min_{\geq_a} \{\mu_a, \nu_a\}$. Without loss of generality, assume $\lambda_a = \mu_a$. Since $\lambda_i = \max_i \{\mu_i, \nu_i\}$, $\lambda_i \; R_i \; \mu_i$. Therefore, $\lambda'_i P_i \mu_i$. Since $\lambda'_i = aP_i\mu_i$ and $i \succ_a j \in \mu_a$, i blocks μ with λ', a contradiction since $\mu \in \pi(B^n)$.

Similarly, suppose i blocks τ with $\tau' \in B^n$. Let $a = \tau'_i$. By definition, $aP_i\tau_i$, and there exists a $j \in \tau_a$ be such that $i \succ_a j$. Without loss of generality, assume $\tau_i = \mu_i$. Since $\tau_i \neq a$, $\tau_a \neq \mu_a$. By Lemma 6, $\tau_a = \nu_a$ and $\nu_a >_a \mu_a$. Therefore, either $j \in \mu_a$ or else $j \in \nu_a \setminus \mu_a$. If $j \in \mu_a$, then i blocks μ with τ' since $aP_i\mu_i$ and $i \succ_a j$. If $j \in \nu_a \setminus \mu_a$, then by the Rural Hospital Theorem, there exists a $k \in \mu_a \setminus \nu_a$. But since $\nu_a >_a \mu_a$, $j \succ_a k$. Therefore, $i \succ_a k$. Since $aP_i\mu_i$, and $i \succ_a k$ and $\mu_k = a$, i blocks μ with τ'. Therefore, in either case, i blocks μ with τ'. This is a contradiction since $\mu \in \pi(B^n)$. □

There are a number of interesting conclusions that are an immediate consequence of the Lattice Theorem. We now know that there is a student optimal (and a student pessimal) fair assignment. Moreover, we can conclude that there is at most one Pareto efficient fair assignment. Otherwise, if μ and ν are both Pareto efficient, and contained in a fair set of assignments, then $\mu \vee \nu$ would be a well defined assignment that Pareto dominated both. Later, we will demonstrate that there exists a Pareto efficient fair assignment.

Corollary 1. *There exists at most one Pareto efficient fair assignment.*

[13] By definition, B^n are the assignments that are not blocked by an assignment in S^n. Since $S^n \subseteq B^n$, no assignment in S^n blocks an assignment in S^n.

[14] By definition, S^{n+1} are the assignments not blocked by an assignment in B^n. Therefore, the result follows from the fact that $S^{n+1} = S^n$.

[15] As discussed in Ehlers and Morrill (2017), this assumption has technical implications. In particular, the Strong Rural Hospital Theorem no longer holds. However, otherwise, the same structure and conclusion for the fair set of assignments continue to hold.

4 Relationship to the Efficiency Adjusted Deferred Acceptance Algorithm

Kesten (2010) introduces a mechanism for the school assignment problem: the Efficiency Adjusted Deferred Acceptance Mechanism (hereafter EADAM). Kesten identifies the source of DA's inefficiency, called interrupter students, and resolves this inefficiency by introducing EADAM. For the precise formulation of the mechanism, we refer the reader to Kesten (2010). EADAM is a subtle and complicated mechanism, but Kesten proves that 1) a student is never harmed by consenting to having her priority waived; 2) EADAM Pareto dominates DA; and 3) if all students consent, then EADAM is Pareto-efficient.[16]

In this section, we prove that the assignment made by EADAM when all students consent is a fair assignment. Since the EADAM assignment is Pareto efficient and a fair set of assignments is a lattice, this proves that EADAM Pareto dominates any other fair assignment. This characterization is analogous to Gale and Shapley's (1962) characterization of DA: the DA assignment is each man's favorite stable assignment. Our result demonstrates that the EADAM assignment is each student's favorite fair assignment.[17]

We will use the simplified EADAM mechanism (hereafter sEADA) introduced by Tang and Yu (2014). The key part of sEADA is the concept of an underdemanded school. For a given assignment μ, a school a is *underdemanded* if for every student i, $\mu_i \ R_i \ a$. There are several facts about underdemanded schools that are critical for Tang and Yu's mechanism. First, under the DA

[16]Two recent papers have established some of the properties of EADAM. Dur and Morrill (2016) demonstrates that EADAM only Pareto improves DA when students submit the same preferences. When students are strategic, they provide an example in which at least one student is made worse off relative to DA in every Nash equilibrium. A corollary of this result is that, in equilibrium, a student may be harmed by consenting. Dur, Gitmez, and Yilmaz (2015) provides the first characterization of EADAM. They prove that it is the unique mechanism that is partially fair, constrained efficient, and gives each student the incentive to consent.

[17]For one-to-one matching problems, this characterization in terms of vNM stability was already known. Bando (2014) proves that EADA produces the male-optimal match in the unique vNM stable set for a marriage problem. For the one-to-one problem, vNM stability and fairness are equivalent. However, for the general problem that we consider, they are not. Moreover, to the best of our knowledge, there are no known results for vNM stability in the general problem. Prior to our work, it was thought that there was no connection between vNM stable sets in one-to-one matching problems and vNM stable sets in many-to-one problems is a college admissions problem (Ehlers, 2005).

assignment, there is always an underdemanded school. For example, the last school that any student applies to is an underdemanded school. Second, a student assigned by DA to an underdemanded school cannot be part of a Pareto improvement. Using these facts, Tang and Yu define sEADA iteratively as follows:[18]

The simplified Efficiency Adjusted Deferred Acceptance Mechanism (sEADA)

Round 0: Run DA on the full population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Round k: Run DA on the remaining population. For each student i assigned to an underdemanded school a, assign i to a; remove i; and reduce a's capacity by one.

Tang and Yu (2014) prove that sEADA and Kesten's EADAM make the same assignment. We will prove that the sEADA assignment is a fair assignment and therefore Pareto dominates any other fair assignment. The following example provides the intuition for this result.

Example 4.

R_i	R_j	R_k	R_l	$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
b	a	a	b	i	j	k	l
a	c	c	d	k	l		
	b			j	i		

In Round 0 of sEADA, the DA assignment is:

$$\begin{pmatrix} i & j & k & l \\ a & b & c & d \end{pmatrix}$$

Note that i envies j's assignment. j envies i and k's assignment, and k envies i's assignment. However, no student strictly prefers d to her assignment. Therefore, d is an underdemanded school. sEADA assigns l to d and removes both the student and the school. Now the assignment problem is:

R_i	R_j	R_k	$\succ_a$	$\succ_b$	$\succ_c$
b	a	a	i	j	k
a	c	c	k	i	
	b		j		

[18] To be precise, this is the definition of sEADA when all students consent to allowing their priority to be violated.

The DA assignment of this problem (and therefore the Round 1 sEADA assignment) is:

$$\begin{pmatrix} i & j & k \\ b & c & a \end{pmatrix}$$

Student l has justified envy of the Round 1 sEADA assignment. Specifically, bP_ld and $l \succ_b i$. However, l cannot be part of a Pareto improvement of the Round 0 assignment since she is assigned to an underdemanded school. The consequence is that in any assignment where l is assigned to b, a student is made worse off relative to the Round 0 assignment. One can verify that this student blocks the new assignment with the DA assignment. Therefore, l has justified envy of b, but there is no fair assignment where she receives b. Therefore, the Round 1 assignment is fair as l's "envy" is "petty".

In the Round 1 assignment, the underdemanded schools are b and c. After removing i and j, all that remains is k and a; therefore, the assignment problem is trivial. The Round 2 sEADA assignment is to assign k to a. Therefore, the final sEADA assignment is:[19]

$$\begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

A key point is that a student assigned to an underdemanded school cannot be part of a Pareto improvement.

Fact: (Lemma 1, Tang and Yu 2014) At the DA matching, no student matched with an underdemanded school is Pareto improvable[20].

This implies that if DA assigns i to an underdemanded school and i prefers school a to her DA assignment, then there is no fair assignment where she is assigned to school a.

[19] Note that this example also demonstrates that the set of α-equitable assignments, as defined by Alcalde and Romero (2015), is not a fair set of assignments. There are two Pareto improvements of the DA assignment:

$$\mu = \begin{pmatrix} i & j & k & l \\ b & a & c & d \end{pmatrix} \qquad \nu = \begin{pmatrix} i & j & k & l \\ b & c & a & d \end{pmatrix}$$

Both μ and ν are α-equitable since they both Pareto improve the DA assignment. ν blocks μ (k prefers ν and has higher priority at a than does j); therefore, the set of α-equitable assignments is not a fair set of assignments.

[20] Student i is Pareto improvable at assignment μ if there exists an assignment ν such that ν is a Pareto improvement of μ and $\nu_i \neq \mu_i$

Lemma 7. *Let $\mu = DA(P)$ and suppose μ_i is underdemanded. Then for any individually rational and nonwasteful assignment ν such that $\nu_i P_i \mu_i$, ν is blocked by μ.*

Proof. The key point is that an underdemanded students cannot be part of a Pareto improvement to μ. Let ν be any assignment such that $\nu_i P_i \mu_i$. ν does not block μ since μ has not justified envy. Suppose for contradiction that μ does not block ν. Let $\overline{\mu}$ and $\overline{\nu}$ be consistent seat assignments, and consider i's $(\overline{\mu}, \overline{\nu})$-cycle. By the General Decomposition Lemma, since $\nu_i P_i \mu_i$, $\nu_j P_j \mu_j$ for every student j in the cycle. Therefore, reassigning the students in the cycle to their assignment under ν is a Pareto improvement. However, this is a contradiction since an underdemanded student cannot be part of a Pareto improvement of μ. $\square$

Theorem 6. *The EADA assignment is a fair assignment.*

Proof. An equivalent way of defining sEADA is as follows. Set $P^1 = P$, and let μ^1 be $DA(P)$. Define U^1 to be the underdemanded students under assignment μ^1. We define preferences P^2 as follows. If $i \in U^1$, then move μ_i^1 to the top of i's preference list. If $i \notin U^1$, then leave i's preferences unchanged. In general, given P^k, we define $\mu_i^k = DA(P^k)$. We define U^k to be the underdemanded students under μ^k, and we modify P^k to create P^{k+1} as follows: if $i \in U^k$, then move μ_i^k to the top of i's preferences; otherwise, leave i's preferences unchanged. Note that $U^k \subseteq U^{k+1}$ since if a student does not envy school a under her true preferences, then the student does not envy school a when we move her assignment to the top of her preference list. The process stops once all students are underdemanded. It is straightforward to verify that this is equivalent to the sEADA procedure.

We prove by induction that for each integer k, (a) $\mu^k \in S^k$ and (b) if $i \in U^k$ and ν is an assignment such that $\nu_i P_i \mu_i^k$, then $\nu \notin B^k$ (note, we set $U^0 = \emptyset$). μ^1 is the DA assignment which has no justified envy. Therefore $\mu^1 \in S^1$. Lemma 7 establishes part (b) of the base step.

Now suppose $k > 1$. Note that if $U^k = U^{k-1}$ then $\mu^k = \mu^{k-1}$ and the result holds trivially. Now suppose $U^k \setminus U^{k-1} \neq \emptyset$. First, we show that $\mu^k \in S^k$. If i has justified envy of μ^k under preferences P_i, then $P_i^k \neq P_i$ (since μ^k has no justified envy under preferences P^k). Therefore, by construction $i \in U^{k-1}$ and $\mu_i^k = \mu_i^{k-1}$. However, if $i \in U^{k-1}$ and ν is an assignment such that $\nu_i P_i \mu_i^k = \mu_i^{k-1}$, then by the inductive hypothesis, $\nu \notin B^{k-1}$. Therefore, $\mu^k \in \pi(B^{k-1}) = S^k$. Now suppose $i \in U^k$. If $i \in U^{k-1}$ then the result follows from the inductive hypothesis. Otherwise, $i \in U^k \setminus U^{k-1}$. Since i is a "new" underdemanded student, we have not yet modified her preferences (we will do so in the next round). In particular, $P_i^k = P_i$. Therefore, Lemma 7 applies. Lemma 7 says that if i is an underdemanded student, and ν is an assignment that i prefers to $\mu^k = DA(P^k)$ (relative to preferences P_i^k; specifically if $\nu_i P_i^k \mu_i^k$), then the DA assignment blocks ν (using preferences P_i^k). To complete our proof, we must show that if ν is an assignment such that $\nu_i P_i \mu_i^k$, then $\nu \notin B^k$. Since $P_i = P_i^k$, $\nu_i P_i^k \mu_i^k$. But by Lemma 7 since i is assigned to an underdemanded school (under preferences P^k) by μ^k, if ν is an assignment such that $\nu_i P_i^k \mu_i^k$, then by Lemma 7, ν is blocked by μ^k. Since $\mu^k \in S^k$, $\nu \notin \pi(S^k) = B^k$.

Since each $S^k \subset S^n$, this proves that the sEADA assignment is in S^n and therefore is fair. □

We now formally state the main conclusion of the paper.

Corollary 2. *The EADA assignment is the unique fair and efficient assignment.*

Proof. This is an immediate corollary of Theorem 6 and Corollary 1. □

5 Conclusion

We provide a well-defined and unique way of determining which school assignments are possible and which are impossible. Essentially, honoring a student's priority at a school implicitly imposes a constraint on the designer: if a student has even higher priority, then this priority should also be honored. We determine when this constraint is impossible to satisfy. We argue that this is critical for determining which assignments are fair as envy of a school is only legitimate if it is possible to assign the student to that school. If it is impossible, we interpret this envy as being petty. Under this interpretation, there is no longer a tension between making a fair assignment and an efficient assignment. Combined, these results suggest that the assignment made by Kesten's EADA is the ideal school assignment. It is the unique assignment that fair and Pareto efficient.

6 Appendix

Here we prove Theorem 1: at the conclusion of the iterative process every assignment has been determined to be possible or impossible. As a reminder, for any set of assignments A, $\pi(A)$ are the assignments not blocked by any assignment in A. We defined S^1 to be the set of assignments with no justified envy. In general, $B^k = \pi(S^k)$ and $S^{k+1} = \pi(B^k)$. We showed that $S^1 \subseteq \ldots \subseteq S^n \subseteq B^n \subseteq \ldots \subseteq B^1$. We defined n as any integer such that $S^n = S^{n+1}$. We also proved that S^n is a Lattice and that there exists a Pareto efficient assignment in S^n (specifically Kesten's EADA assignment).

Theorem 1: $S^n = B^n$.

Proof. Suppose for contradiction that there exists an assignment $\nu \in B^n \backslash S^n$. Since $\nu \notin S^{n+1} = S^n$, ν is blocked by some student i with assignment $\mu \in B^n$. Let $a = \mu_i$. Note that there does not exist an assignment $\phi \in S^n$ such that $\phi_i = a$. Otherwise, i would block ν with ϕ (in which case $\nu \notin B^n$). Since S^n is a lattice, there is a well-defined $\mu^{\max}$ and $\mu^{\min}$ (relative to the order $>$). We demonstrated in Theorem 6 that $\mu^{\max}$ is the assignment made by EADA. A symmetric argument demonstrates that $\mu^{\min}$ is the assignment made by the school-proposing EADA mechanism.[21]

Claim 1: $\mu_i^{\max}\ P_i\ \mu_i\ P_i\ \mu_i^{\min}$

In the proof of Theorem 6, we demonstrated the following: for any student j, if b is j's assignment under EADA and c is a school such that cP_jb, then there does not exist an assignment $\lambda \in B^n$ such that $\lambda_j = c$. $\mu_i^{\max} \neq a = \mu_i$ (since $\mu_i^{\max} \in S^n$ and there does not exist a $\phi \in S^n$ such that $\phi_i = a$).Since $\mu \in B^n$, it follows that $\mu_i^{\max} P_i \mu_i$. A symmetric argument yields that i strictly prefers a to her assignment under the school proposing EADA as any assignment in which i receives a worse assignment than under the school proposing EADA is blocked by the school-proposing EADA assignment.

As a result of Claim 1, both $\{\phi \in S^n | \phi_i P_i a\}$ and $\{\phi \in S^n | a P_i \phi_i\}$ are non-empty. Since S^n is a lattice, the following are well defined.

$$\overline{\mu} := \min_{>} \{\phi \in S^n | \phi_i P_i a\} \tag{5}$$

[21]To the best of our knowledge, the school-proposing EADA has not been defined in the literature; however, when preferences are responsive, it is straightforward to do so. Specifically, create q_a clones of each school a. Each clone has the same priorities over students as does a. Each student prefers lower numbered clones to higher numbered clones. Run sEADA with the clones proposing to students.

and

$$\underline{\mu} := \max_{>} \{\phi \in S^n | aP_i\phi_i\} \tag{6}$$

Claim 2: For any $\phi \in S^n$, either $\phi R\overline{\mu}$ or else $\underline{\mu}R\phi$.

This is an immediate consequence of the fact that $\phi_i \neq a$ for any $\phi \in S^n$. Therefore, either $\phi_i P_i a$, in which case $\phi R\overline{\mu}$ or else $aP_i\phi_i$ in which case $\underline{\mu}R\phi$.

Let $k = \min_{\succ_a} \overline{\mu}_a$ (the lowest priority student assigned to a by $\overline{\mu}$). Consider the following generalization of the Deferred Acceptance algorithm. For each student l, define all schools that l strictly prefers to $\overline{\mu}_l$ to have rejected l. Next, we reject k from a. This begins what we will refer to as the vacancy chain. We only allow a student l to apply to a school b if there exists a $\phi \in B^n$ such that $\phi_l = b$ (we call such a school **achievable** for the student). We have k apply to her favorite achievable school that has not yet rejected her. Each time a school receives a new application, if it has an available seat, then the school accepts the application and does not reject a student. If it does not have an available seat, then it rejects the lowest ranked student. Each time a student is rejected, it applies to her next favorite achievable school. The process ends when there is no new rejection. Let ϕ be the assignment that results from this process.

Claim 3: $\phi R\underline{\mu}$.

If not, then let l be the first student in the vacancy chain rejected by her assignment under $\underline{\mu}$. By the Rural Hospital Theorem, when $\underline{\mu}_l$ rejects l, it must be holding onto the application from a student m such that $\underline{\mu}_m \neq \underline{\mu}_l$. m has not been rejected by $\underline{\mu}_m$ since l is the first student rejected by her assignment under $\underline{\mu}$. Since m applied to $\underline{\mu}_l$ before applying to $\underline{\mu}_m$, by revealed preference $\underline{\mu}_l P_m \underline{\mu}_m$. Since $\underline{\mu}_l$ rejected l in favor of m, $m \succ_{\underline{\mu}_l} l$. Since m can only apply to an achievable school, there exists a $\phi \in B^n$ such that $\phi_m = \underline{\mu}_l$. Therefore, m blocks $\underline{\mu}$ with ϕ, contradicting the fact that $\underline{\mu} \in S^{n+1}$ and therefore is not blocked by any assignment in B^n.

Claim 4: the vacancy chain concludes when the first time a student applies to a.

There are only three possible ways for the vacancy chain to end: 1) a student applies to a, 2) a student applies to a school b and $|\overline{\mu}_b| < q_b$, and 3) a student is rejected by all of her achievable schools. If the third case occurred, then there would exist a student l such that $\overline{\mu}_l \neq \emptyset$, but $\phi_l = \emptyset$. Since $\overline{\mu}_l \neq \emptyset$, $\underline{\mu}_l \neq \emptyset$ by the Rural Hospital Theorem. Since $\phi_l \neq \underline{\mu}_l$ and $\phi_l R_l \underline{\mu}_l$, $\phi_l P_l \underline{\mu}_l$. But this contradicts the individual rationality of $\underline{\mu}$ since $\phi_l = \emptyset$. Similarly, if the second case occurred, then for some student l, $\phi_l = b$ and $|\overline{\mu}_b| < q_b$. By the Strong Rural Hospital Theorem, $\overline{\mu}_b = \underline{\mu}_b$. Since $\phi_l R_l \underline{\mu}_l$ and $b \neq \underline{\mu}_l$, $bP_l\underline{\mu}_l$. But this implies that $\underline{\mu}$ is wasteful as b has available seats under $\underline{\mu}$ which is a contradiction. Therefore, only the first case is possible. The vacancy chain concludes the first time a student applies to a.

Let l be the student who ends the vacancy chain. Note that $\phi_a = \overline{\mu}_a \cup \{l\} \setminus \{k\}$.

Claim 5: All schools have higher ranked students under ϕ than under $\overline{\mu}$ ($\phi > \overline{\mu}$).

This is clear for all schools $b \neq a$ since such a school b only rejects a student in favor of a higher ranked applicant. We now consider school a. As a reminder, $\phi_a = \overline{\mu}_a \cup \{l\} \setminus \{k\}$. Therefore, we must show that $l \succ_a k$. a is achievable for l, so there exists a $\sigma \in B^n$ such that $\sigma_l = a$. Since $\overline{\mu} \in S^n$ and $\sigma \in B^n$, μ and σ do not block each other. Since $\phi_a = \overline{\mu}_a \cup \{l\} \setminus \{k\}$, l and k are in the same cycle. Since $\overline{\mu}_l P_l a = \sigma_l$, by the General Decomposition Lemma, $l \succ_a k$. So indeed, for every school b, $\phi_b \geq_b \overline{\mu}_b$.

Claim 6: $\phi \in S^n$.

Suppose for contradiction that a student j blocks ϕ with $\nu \in B^n$. Let $\nu_j = b$. We know that $bP_j\phi_j$ and the j has higher priority than some student $k \in \phi_b$. The key point is that $\phi_b \geq_b \overline{\mu}_b$ (b has higher ranked students under ϕ than under $\overline{\mu}$). We can immediately concluded that $\overline{\mu}_j \neq b$ and that j has higher priority than at least one student assigned to b by $\overline{\mu}$. Note that if $bP_j\overline{\mu}_j$, j also blocks $\overline{\mu}$ with ν (a contradiction as $\overline{\mu}$ is unblocked). Therefore, it must be that that $\overline{\mu}_j P_j \nu_j P_j \phi_j$. But this means that j was part of the vacancy chain, and in particular, that j applied to ν_j (ν_j is achievable). However, j was rejected by b and k was not which is a contradiction since $j \succ_b k$.

Claim 7: $\phi = \underline{\mu}$

$\overline{\mu}_k = aP_k\phi_k$. Therefore, it cannot be that $\phi R \overline{\mu}$. Since $\phi \in S^n$, it follows from Claim 2 that $\underline{\mu} R \phi$. Since, $\phi R \underline{\mu}$ by Claim 2, we conclude that $\phi = \underline{\mu}$.

However, this is a contradiction. The vacancy chain concludes the first time a student applies to a. For student i, $\overline{\mu}_i P_i a P_i \underline{\mu} = \phi$. Therefore, i participated in the vacancy chain. a is achievable for i and i strictly prefers a to ϕ_i. Therefore, i must have applied to a at some point in the vacancy chain and must have been rejected by a. However, this is a contradiction as the vacancy chain stops the first time a student applies to a. □

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